1 ☐ FUNCTIONS AND LIMITS

1.1 Four Ways to Represent a Function

1. The functions $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$ give exactly the same output values for every input value, so f and g are equal.

2. $f(x) = \dfrac{x^2 - x}{x - 1} = \dfrac{x(x - 1)}{x - 1} = x$ for $x - 1 \neq 0$, so f and g [where $g(x) = x$] are not equal because $f(1)$ is undefined and $g(1) = 1$.

3. (a) The point $(-2, 2)$ lies on the graph of g, so $g(-2) = 2$. Similarly, $g\,(0) = -2$, $g\,(2) = 1$, and $g\,(3) \approx 2.5$.

(b) Only the point $(-4, 3)$ on the graph has a y-value of 3, so the only value of x for which $g(x) = 3$ is -4.

(c) The function outputs $g(x)$ are never greater than 3, so $g(x) \leq 3$ for the entire domain of the function. Thus, $g(x) \leq 3$ for $-4 \leq x \leq 4$ (or, equivalently, on the interval $[-4, 4]$).

(d) The domain consists of all x-values on the graph of g: $\{x \mid -4 \leq x \leq 4\} = [-4, 4]$. The range of g consists of all the y-values on the graph of g: $\{y \mid -2 \leq y \leq 3\} = [-2, 3]$.

(e) For any $x_1 < x_2$ in the interval $[0, 2]$, we have $g(x_1) < g(x_2)$. [The graph rises from $(0, -2)$ to $(2, 1)$.] Thus, $g(x)$ is increasing on $[0, 2]$.

4. (a) From the graph, we have $f(-4) = -2$ and $g(3) = 4$.

(b) Since $f(-3) = -1$ and $g(-3) = 2$, or by observing that the graph of g is above the graph of f at $x = -3$, $g(-3)$ is larger than $f(-3)$.

(c) The graphs of f and g intersect at $x = -2$ and $x = 2$, so $f(x) = g(x)$ at these two values of x.

(d) The graph of f lies below or on the graph of g for $-4 \leq x \leq -2$ and for $2 \leq x \leq 3$. Thus, the intervals on which $f(x) \leq g(x)$ are $[-4, -2]$ and $[2, 3]$.

(e) $f\,(x) = -1$ is equivalent to $y = -1$, and the points on the graph of f with y-values of -1 are $(-3, -1)$ and $(4, -1)$, so the solution of the equation $f(x) = -1$ is $x = -3$ or $x = 4$.

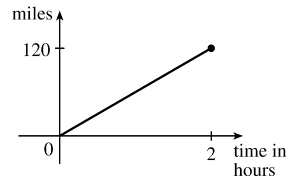
(f) For any $x_1 < x_2$ in the interval $[-4, 0]$, we have $g(x_1) > g(x_2)$. Thus, $g(x)$ is decreasing on $[-4, 0]$.

(g) The domain of f is $\{x \mid -4 \leq x \leq 4\} = [-4, 4]$. The range of f is $\{y \mid -2 \leq y \leq 3\} = [-2, 3]$.

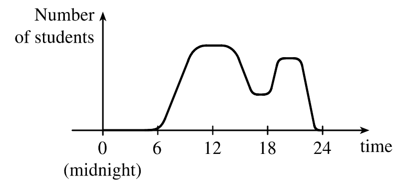
(h) The domain of g is $\{x \mid -4 \leq x \leq 3\} = [-4, 3]$. Estimating the lowest point of the graph of g as having coordinates $(0, 0.5)$, the range of g is approximately $\{y \mid 0.5 \leq y \leq 4\} = [0.5, 4]$.

5. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq a \leq 115$. Written in interval notation, the range is $[-85, 115]$.

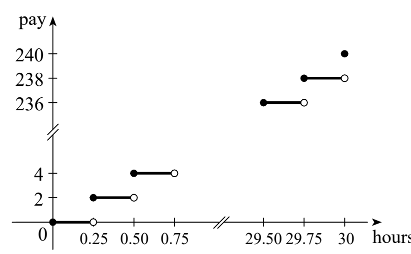
6. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \le t \le 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \le d \le 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \le t \le 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \le N \le k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid $8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



7. We solve $3x - 5y = 7$ for y: $3x - 5y = 7 \quad \Leftrightarrow \quad -5y = -3x + 7 \quad \Leftrightarrow \quad y = \frac{3}{5}x - \frac{7}{5}$. Since the equation determines exactly one value of y for each value of x, the equation defines y as a function of x.

8. We solve $3x^2 - 2y = 5$ for y: $3x^2 - 2y = 5 \quad \Leftrightarrow \quad -2y = -3x^2 + 5 \quad \Leftrightarrow \quad y = \frac{3}{2}x^2 - \frac{5}{2}$. Since the equation determines exactly one value of y for each value of x, the equation defines y as a function of x.

9. We solve $x^2 + (y-3)^2 = 5$ for y: $x^2 + (y-3)^2 = 5 \quad \Leftrightarrow \quad (y-3)^2 = 5 - x^2 \quad \Leftrightarrow \quad y - 3 = \pm\sqrt{5 - x^2} \quad \Leftrightarrow$ $y = 3 \pm \sqrt{5 - x^2}$. Some input values x correspond to more than one output y. (For instance, $x = 1$ corresponds to $y = 1$ and to $y = 5$.) Thus, the equation does *not* define y as a function of x.

10. We solve $2xy + 5y^2 = 4$ for y: $2xy + 5y^2 = 4 \quad \Leftrightarrow \quad 5y^2 + (2x)\,y - 4 = 0 \quad \Leftrightarrow$

$$y = \frac{-2x \pm \sqrt{(2x)^2 - 4(5)(-4)}}{2(5)} = \frac{-2x \pm \sqrt{4x^2 + 80}}{10} = \frac{-x \pm \sqrt{x^2 + 20}}{5}$$

(using the quadratic formula). Some input values x correspond to more than one output y. (For instance, $x = 4$ corresponds to $y = -2$ and to $y = 2/5$.) Thus, the equation does *not* define y as a function of x.

11. We solve $(y+3)^3 + 1 = 2x$ for y: $(y+3)^3 + 1 = 2x \quad \Leftrightarrow \quad (y+3)^3 = 2x - 1 \quad \Leftrightarrow \quad y + 3 = \sqrt[3]{2x - 1} \quad \Leftrightarrow$ $y = -3 + \sqrt[3]{2x - 1}$. Since the equation determines exactly one value of y for each value of x, the equation defines y as a function of x.

12. We solve $2x - |y| = 0$ for y: $2x - |y| = 0 \;\Leftrightarrow\; |y| = 2x \;\Leftrightarrow\; y = \pm 2x$. Some input values x correspond to more than one output y. (For instance, $x = 1$ corresponds to $y = -2$ and to $y = 2$.) Thus, the equation does *not* define y as a function of x.

13. The height 60 in ($x = 60$) corresponds to shoe sizes 7 and 8 ($y = 7$ and $y = 8$). Since an input value x corresponds to more than output value y, the table does *not* define y as a function of x.

14. Each year x corresponds to exactly one tuition cost y. Thus, the table defines y as a function of x.

15. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

16. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

17. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

18. No, the curve is not the graph of a function since for $x = 0$, ± 1, and ± 2, there are infinitely many points on the curve.

19. (a) When $t = 1950$, $T \approx 13.8^\circ\text{C}$, so the global average temperature in 1950 was about 13.8°C.

(b) When $T = 14.2^\circ\text{C}$, $t \approx 1990$.

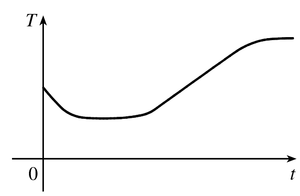
(c) The global average temperature was smallest in 1910 (the year corresponding to the lowest point on the graph) and largest in 2000 (the year corresponding to the highest point on the graph).

(d) When $t = 1910$, $T \approx 13.5^\circ\text{C}$, and when $t = 2000$, $T \approx 14.4^\circ\text{C}$. Thus, the range of T is about $[13.5, 14.4]$.

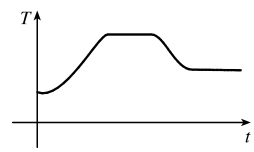
20. (a) The ring width varies from near 0 mm to about 1.6 mm, so the range of the ring width function is approximately $[0, 1.6]$.

(b) According to the graph, the earth gradually cooled from 1550 to 1700, warmed into the late 1700s, cooled again into the late 1800s, and has been steadily warming since then. In the mid-19th century, there was variation that could have been associated with volcanic eruptions.

21. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



22. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.

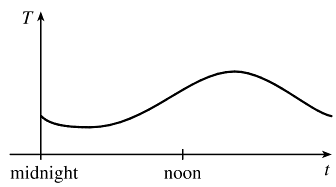


23. (a) The power consumption at 6 AM is 500 MW, which is obtained by reading the value of power P when $t = 6$ from the graph. At 6 PM we read the value of P when $t = 18$, obtaining approximately 730 MW.

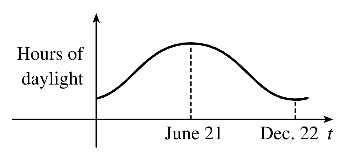
(b) The minimum power consumption is determined by finding the time for the lowest point on the graph, $t = 4$, or 4 AM. The maximum power consumption corresponds to the highest point on the graph, which occurs just before $t = 12$, or right before noon. These times are reasonable, considering the power consumption schedules of most individuals and businesses.

24. Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

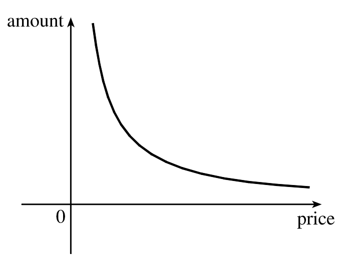
25. Of course, this graph depends strongly on the geographical location!



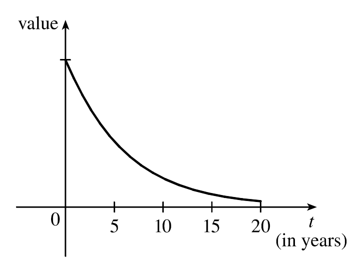
26. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



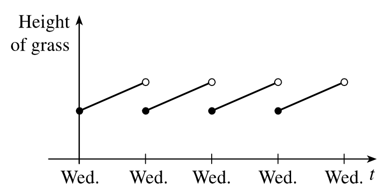
27. As the price increases, the amount sold decreases.



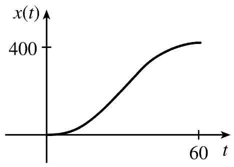
28. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



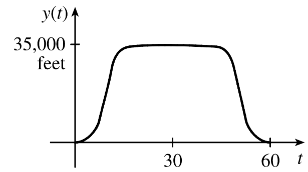
29.



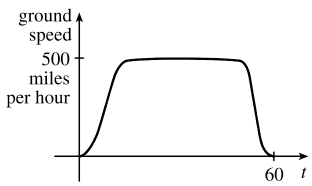
30. (a)



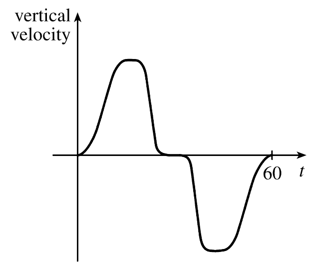
(b)



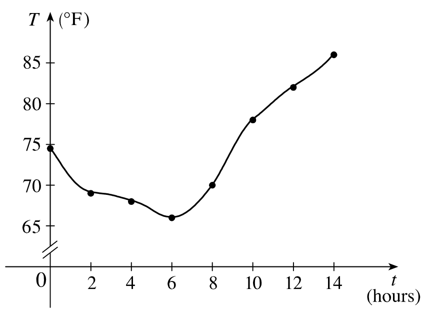
(c)



(d)

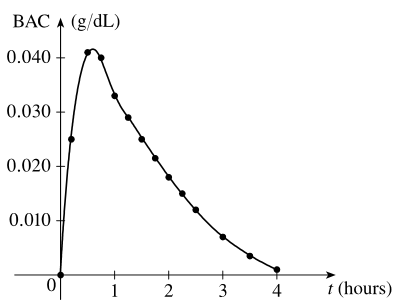


31. (a)



(b) 9:00 AM corresponds to $t = 9$. When $t = 9$, the temperature T is about $74°\text{F}$.

32. (a)



(b) The blood alcohol concentration rises rapidly, then slowly decreases to near zero.

33. $f(x) = 3x^2 - x + 2$.

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12$.

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16$.

$f(a) = 3a^2 - a + 2$.

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2$.

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4$.

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4$.

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2$.

[continued]

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$
$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

34. $g(x) = \dfrac{x}{\sqrt{x+1}}.$

$g(0) = \dfrac{0}{\sqrt{0+1}} = 0.$

$g(3) = \dfrac{3}{\sqrt{3+1}} = \dfrac{3}{2}.$

$5g(a) = 5 \cdot \dfrac{a}{\sqrt{a+1}} = \dfrac{5a}{\sqrt{a+1}}.$

$\dfrac{1}{2}g(4a) = \dfrac{1}{2} \cdot g(4a) = \dfrac{1}{2} \cdot \dfrac{4a}{\sqrt{4a+1}} = \dfrac{2a}{\sqrt{4a+1}}.$

$g(a^2) = \dfrac{a^2}{\sqrt{a^2+1}}$; $[g(a)]^2 = \left(\dfrac{a}{\sqrt{a+1}}\right)^2 = \dfrac{a^2}{a+1}.$

$g(a+h) = \dfrac{(a+h)}{\sqrt{(a+h)+1}} = \dfrac{a+h}{\sqrt{a+h+1}}.$

$g(x-a) = \dfrac{(x-a)}{\sqrt{(x-a)+1}} = \dfrac{x-a}{\sqrt{x-a+1}}.$

35. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$,

and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h.$

36. $f(x) = x^3$, so $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$,

and $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \dfrac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2.$

37. $f(x) = \dfrac{1}{x}$, so $\dfrac{f(x) - f(a)}{x - a} = \dfrac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} = \dfrac{\dfrac{a - x}{xa}}{x - a} = \dfrac{a - x}{xa(x-a)} = \dfrac{-1(x-a)}{xa(x-a)} = -\dfrac{1}{ax}.$

38. $f(x) = \sqrt{x+2}$, so $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{\sqrt{x+2} - \sqrt{3}}{x - 1}$. Depending upon the context, this may be considered simplified.

Note: We may also rationalize the numerator:

$$\frac{\sqrt{x+2} - \sqrt{3}}{x-1} = \frac{\sqrt{x+2} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{x+2} + \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} = \frac{(x+2) - 3}{(x-1)\left(\sqrt{x-2} + \sqrt{3}\right)}$$
$$= \frac{x-1}{(x-1)\left(\sqrt{x-2} + \sqrt{3}\right)} = \frac{1}{\sqrt{x+2} + \sqrt{3}}$$

39. $f(x) = (x+4)/(x^2 - 9)$ is defined for all x except when $0 = x^2 - 9 \;\Leftrightarrow\; 0 = (x+3)(x-3) \;\Leftrightarrow\; x = -3$ or 3, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

40. The function $f(x) = \dfrac{x^2+1}{x^2+4x-21}$ is defined for all values of x except those for which $x^2+4x-21=0 \iff (x+7)(x-3)=0 \iff x=-7$ or $x=3$. Thus, the domain is $\{x \in \mathbb{R} \mid x \neq -7, 3\} = (-\infty,-7) \cup (-7,3) \cup (3,\infty)$.

41. $f(t) = \sqrt[3]{2t-1}$ is defined for all real numbers. In fact $\sqrt[3]{p(t)}$, where $p(t)$ is a polynomial, is defined for all real numbers. Thus, the domain is $\mathbb{R}$, or $(-\infty,\infty)$.

42. $g(t) = \sqrt{3-t} - \sqrt{2+t}$ is defined when $3-t \geq 0 \iff t \leq 3$ and $2+t \geq 0 \iff t \geq -2$. Thus, the domain is $-2 \leq t \leq 3$, or $[-2,3]$.

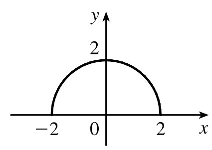
43. $h(x) = 1\big/\sqrt[4]{x^2-5x}$ is defined when $x^2-5x>0 \iff x(x-5)>0$. Note that $x^2-5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x<0$ or $x>5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty,0) \cup (5,\infty)$.

44. $f(u) = \dfrac{u+1}{1+\dfrac{1}{u+1}}$ is defined when $u+1 \neq 0$ $[u \neq -1]$ and $1+\dfrac{1}{u+1} \neq 0$. Since $1+\dfrac{1}{u+1}=0 \iff \dfrac{1}{u+1}=-1 \iff 1=-u-1 \iff u=-2$, the domain is $\{u \mid u \neq -2, u \neq -1\} = (-\infty,-2) \cup (-2,-1) \cup (-1,\infty)$.

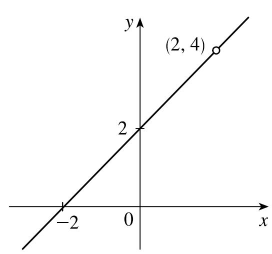
45. $F(p) = \sqrt{2-\sqrt{p}}$ is defined when $p \geq 0$ and $2-\sqrt{p} \geq 0$. Since $2-\sqrt{p} \geq 0 \iff 2 \geq \sqrt{p} \iff \sqrt{p} \leq 2 \iff 0 \leq p \leq 4$, the domain is $[0,4]$.

46. The function $h(x) = \sqrt{x^2-4x-5}$ is defined when $x^2-4x-5 \geq 0 \iff (x+1)(x-5) \geq 0$. The polynomial $p(x) = x^2-4x-5$ may change signs only at its zeros, so we test values of x on the intervals separated by $x=-1$ and $x=5$: $p(-2)=7>0$, $p(0)=-5<0$, and $p(6)=7>0$. Thus, the domain of h, equivalent to the solution intervals of $p(x) \geq 0$, is $\{x \mid x \leq -1 \text{ or } x \geq 5\} = (-\infty,-1] \cup [5,\infty)$.

47. $h(x) = \sqrt{4-x^2}$. Now $y = \sqrt{4-x^2} \Rightarrow y^2 = 4-x^2 \iff x^2+y^2=4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4-x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2,2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0,2]$.

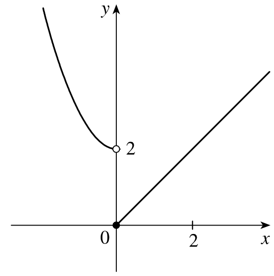


48. The function $f(x) = \dfrac{x^2-4}{x-2}$ is defined when $x-2 \neq 0 \iff x \neq 2$, so the domain is $\{x \mid x \neq 2\} = (-\infty,2) \cup (2,\infty)$. On its domain, $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2} = x+2$. Thus, the graph of f is the line $y = x+2$ with a hole at $(2,4)$.



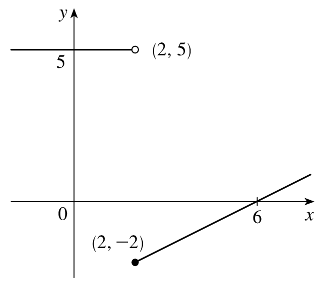
49. $f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$f(-3) = (-3)^2 + 2 = 11$, $f(0) = 0$, and $f(2) = 2$.



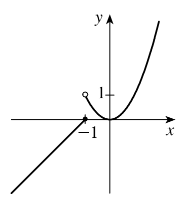
50. $f(x) = \begin{cases} 5 & \text{if } x < 2 \\ \frac{1}{2}x - 3 & \text{if } x \geq 2 \end{cases}$

$f(-3) = 5$, $f(0) = 5$, and $f(2) = \frac{1}{2}(2) - 3 = -2$.



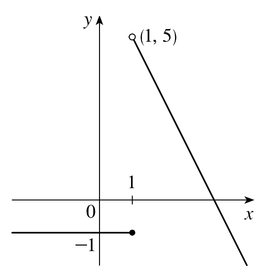
51. $f(x) = \begin{cases} x + 1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

$f(-3) = -3 + 1 = -2$, $f(0) = 0^2 = 0$, and $f(2) = 2^2 = 4$.



52. $f(x) = \begin{cases} -1 & \text{if } x \leq 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

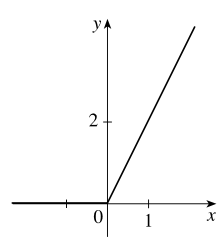
$f(-3) = -1$, $f(0) = -1$, and $f(2) = 7 - 2(2) = 3$.



53. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

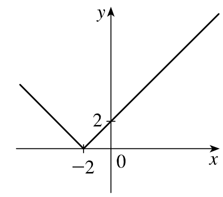
so $f(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



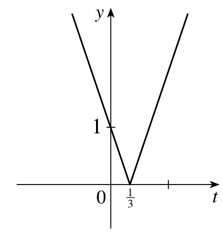
54. $f(x) = |x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases}$

$= \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$



55. $g(t) = |1-3t| = \begin{cases} 1-3t & \text{if } 1-3t \geq 0 \\ -(1-3t) & \text{if } 1-3t < 0 \end{cases}$

$= \begin{cases} 1-3t & \text{if } t \leq \frac{1}{3} \\ 3t-1 & \text{if } t > \frac{1}{3} \end{cases}$

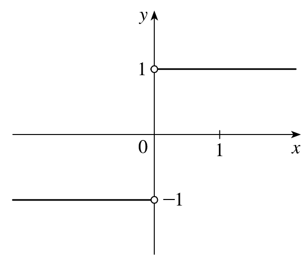


56. $f(x) = \dfrac{|x|}{x}$

The domain of f is $\{x \mid x \neq 0\}$ and $|x| = x$ if $x > 0$, $|x| = -x$ if $x < 0$.

So we can write

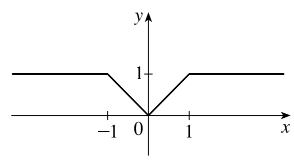
$$f(x) = \begin{cases} \dfrac{-x}{x} = -1 & \text{if } x < 0 \\ \dfrac{x}{x} = 1 & \text{if } x > 0 \end{cases}$$



57. To graph $f(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| > 1 \end{cases}$, graph $y = |x|$ [Figure 16]

for $-1 \leq x \leq 1$ and graph $y = 1$ for $x > 1$ and for $x < -1$.

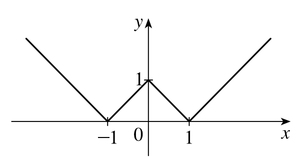
We could rewrite f as $f(x) = \begin{cases} 1 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$.



58. $g(x) = \big||x|-1\big| = \begin{cases} |x|-1 & \text{if } |x|-1 \geq 0 \\ -(|x|-1) & \text{if } |x|-1 < 0 \end{cases}$

$= \begin{cases} |x|-1 & \text{if } |x| \geq 1 \\ -|x|+1 & \text{if } |x| < 1 \end{cases}$

$= \begin{cases} x-1 & \text{if } |x| \geq 1 \text{ and } x \geq 0 \\ -x-1 & \text{if } |x| \geq 1 \text{ and } x < 0 \\ -x+1 & \text{if } |x| < 1 \text{ and } x \geq 0 \\ -(-x)+1 & \text{if } |x| < 1 \text{ and } x < 0 \end{cases} = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x-1 & \text{if } x \leq -1 \\ -x+1 & \text{if } 0 \leq x < 1 \\ x+1 & \text{if } -1 < x < 0 \end{cases}$



59. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of the line segment joining the points $(1, -3)$ and $(5, 7)$ is $\dfrac{7-(-3)}{5-1} = \dfrac{5}{2}$, so an equation is $y - (-3) = \frac{5}{2}(x-1)$. The function is $f(x) = \frac{5}{2}x - \frac{11}{2}$, $1 \le x \le 5$.

60. The slope of the line segment joining the points $(-5, 10)$ and $(7, -10)$ is $\dfrac{-10-10}{7-(-5)} = -\dfrac{5}{3}$, so an equation is $y - 10 = -\frac{5}{3}[x - (-5)]$. The function is $f(x) = -\frac{5}{3}x + \frac{5}{3}$, $-5 \le x \le 7$.

61. We need to solve the given equation for y. $x + (y-1)^2 = 0 \quad\Leftrightarrow\quad (y-1)^2 = -x \quad\Leftrightarrow\quad y - 1 = \pm\sqrt{-x} \quad\Leftrightarrow$ $y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

62. $x^2 + (y-2)^2 = 4 \quad\Leftrightarrow\quad (y-2)^2 = 4 - x^2 \quad\Leftrightarrow\quad y - 2 = \pm\sqrt{4-x^2} \quad\Leftrightarrow\quad y = 2 \pm \sqrt{4-x^2}$. The top half is given by the function $f(x) = 2 + \sqrt{4-x^2}$, $-2 \le x \le 2$.

63. For $0 \le x \le 3$, the graph is the line with slope -1 and y-intercept 3, that is, $y = -x + 3$. For $3 < x \le 5$, the graph is the line with slope 2 passing through $(3, 0)$; that is, $y - 0 = 2(x-3)$, or $y = 2x - 6$. So the function is

$$f(x) = \begin{cases} -x + 3 & \text{if } 0 \le x \le 3 \\ 2x - 6 & \text{if } 3 < x \le 5 \end{cases}$$

64. For $-4 \le x \le -2$, the graph is the line with slope $-\frac{3}{2}$ passing through $(-2, 0)$; that is, $y - 0 = -\frac{3}{2}[x - (-2)]$, or $y = -\frac{3}{2}x - 3$. For $-2 < x < 2$, the graph is the top half of the circle with center $(0, 0)$ and radius 2. An equation of the circle is $x^2 + y^2 = 4$, so an equation of the top half is $y = \sqrt{4-x^2}$. For $2 \le x \le 4$, the graph is the line with slope $\frac{3}{2}$ passing through $(2, 0)$; that is, $y - 0 = \frac{3}{2}(x-2)$, or $y = \frac{3}{2}x - 3$. So the function is

$$f(x) = \begin{cases} -\frac{3}{2}x - 3 & \text{if } -4 \le x \le -2 \\ \sqrt{4-x^2} & \text{if } -2 < x < 2 \\ \frac{3}{2}x - 3 & \text{if } 2 \le x \le 4 \end{cases}$$

65. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

66. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

67. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

68. Let the length, width, and height of the closed rectangular box be denoted by L, W, and H, respectively. The length is twice the width, so $L = 2W$. The volume V of the box is given by $V = LWH$. Since $V = 8$, we have $8 = (2W)WH \quad \Rightarrow \quad 8 = 2W^2H \quad \Rightarrow \quad H = \dfrac{8}{2W^2} = \dfrac{4}{W^2}$, and so $H = f(W) = \dfrac{4}{W^2}$.

69. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

70. Let r and h denote the radius and the height of the right circular cylinder, respectively. Then the volume V is given by $V = \pi r^2 h$, and for this particular cylinder we have $\pi r^2 h = 25 \quad \Leftrightarrow \quad r^2 = \dfrac{25}{\pi h}$. Solving for r and rejecting the negative solution gives $r = \dfrac{5}{\sqrt{\pi h}}$, so $r = f(h) = \dfrac{5}{\sqrt{\pi h}}$ in.

71. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so
$V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$.
The sides L, W, and x must be positive. Thus, $L > 0 \quad \Leftrightarrow \quad 20 - 2x > 0 \quad \Leftrightarrow \quad x < 10$;
$W > 0 \quad \Leftrightarrow \quad 12 - 2x > 0 \quad \Leftrightarrow \quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

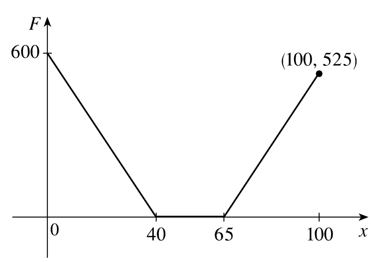
72. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window.
The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad \Leftrightarrow \quad 2h = 30 - x - \frac{1}{2}\pi x \quad \Leftrightarrow \quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2 = 15x - \frac{4}{8}x^2 - \frac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi + 4}{8}\right).$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad \Leftrightarrow \quad 30 - x - \frac{1}{2}\pi x > 0 \quad \Leftrightarrow \quad 60 > 2x + \pi x \quad \Leftrightarrow \quad x < \dfrac{60}{2 + \pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2 + \pi}$.

73. We can summarize the amount of the fine with a piecewise defined function.

$$F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \le x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$$



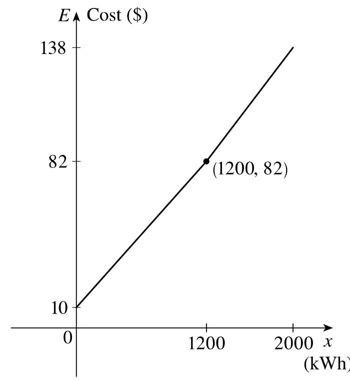
74. For the first 1200 kWh, $E(x) = 10 + 0.06x$.

For usage over 1200 kWh, the cost is

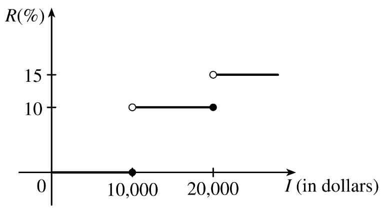
$E(x) = 10 + 0.06(1200) + 0.07(x - 1200) = 82 + 0.07(x - 1200)$.

Thus,

$$E(x) = \begin{cases} 10 + 0.06x & \text{if } 0 \leq x \leq 1200 \\ 82 + 0.07(x - 1200) & \text{if } x > 1200 \end{cases}$$



75. (a)

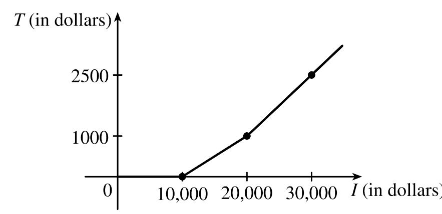


(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \$400$.

On \$26,000, tax is assessed on \$16,000, and

$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



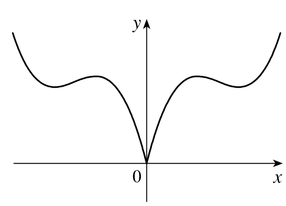
76. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

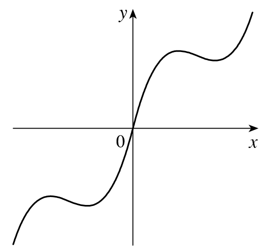
77. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

78. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

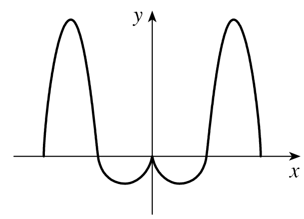
79. (a) The graph of an even function is symmetric about the y-axis. We reflect the given portion of the graph of f about the y-axis in order to complete it.



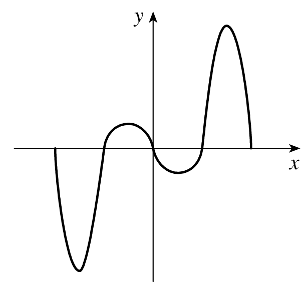
(b) For an odd function, $f(-x) = -f(x)$. The graph of an odd function is symmetric about the origin. We rotate the given portion of the graph of f through $180°$ about the origin in order to complete it.



80. (a) The graph of an even function is symmetric about the y-axis. We reflect the given portion of the graph of f about the y-axis in order to complete it.



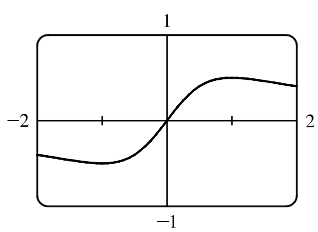
(b) The graph of an odd function is symmetric about the origin. We rotate the given portion of the graph of f through $180°$ about the origin in order to complete it.



81. $f(x) = \dfrac{x}{x^2+1}$.

$f(-x) = \dfrac{-x}{(-x)^2+1} = \dfrac{-x}{x^2+1} = -\dfrac{x}{x^2+1} = -f(x)$.

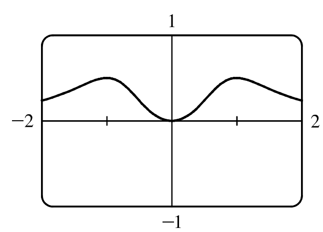
Since $f(-x) = -f(x)$, f is an odd function.



82. $f(x) = \dfrac{x^2}{x^4+1}$.

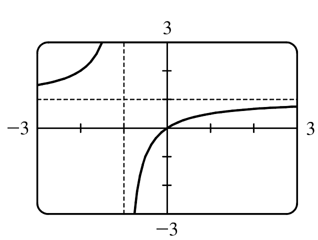
$f(-x) = \dfrac{(-x)^2}{(-x)^4+1} = \dfrac{x^2}{x^4+1} = f(x)$.

Since $f(-x) = f(x)$, f is an even function.



83. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

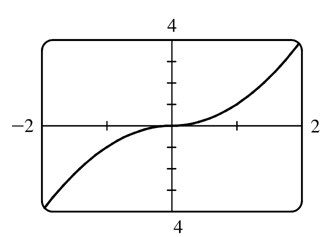
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



84. $f(x) = x\,|x|$.

$f(-x) = (-x)\,|-x| = (-x)\,|x| = -(x\,|x|)$
$= -f(x)$

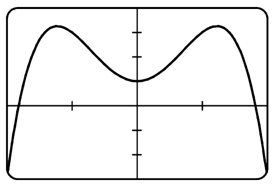
Since $f(-x) = -f(x)$, f is an odd function.



85. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1+3(-x)^2-(-x)^4 = 1+3x^2-x^4 = f(x)$.

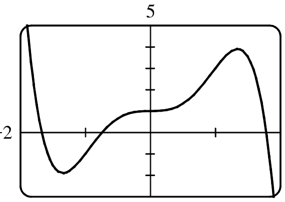
Since $f(-x) = f(x)$, f is an even function.



86. $f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5)$$
$$= 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



87. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an *odd* function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is *neither* even nor odd. (Exception: if f is the zero function, then $f+g$ will be *odd*. If g is the zero function, then $f+g$ will be *even*.)

88. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = x^3 + 3x^2$ is a polynomial function of degree 3. (This function is also an algebraic function.)

(b) $g(t) = \cos^2 t - \sin t$ is a trigonometric function.

(c) $r(t) = t^{\sqrt{3}}$ is a power function.

(d) $v(t) = 8^t$ is an exponential function.

(e) $y = \dfrac{\sqrt{x}}{x^2 + 1}$ is an algebraic function. It is the quotient of a root of a polynomial and a polynomial of degree 2.

(f) $g(u) = \log_{10} u$ is a logarithmic function.

2. (a) $f(t) = \dfrac{3t^2 + 2}{t}$ is a rational function. (This function is also an algebraic function.)

(b) $h(r) = 2.3^r$ is an exponential function.

(c) $s(t) = \sqrt{t+4}$ is an algebraic function. It is a root of a polynomial.

(d) $y = x^4 + 5$ is a polynomial function of degree 4.

(e) $g(x) = \sqrt[3]{x}$ is a root function. Rewriting $g(x)$ as $x^{1/3}$, we recognize the function also as a power function. (This function is, further, an algebraic function because it is a root of a polynomial.)

(f) $y = \dfrac{1}{x^2}$ is a rational function. Rewriting y as x^{-2}, we recognize the function also as a power function. (This function is, further, an algebraic function because it is the quotient of two polynomials.)

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

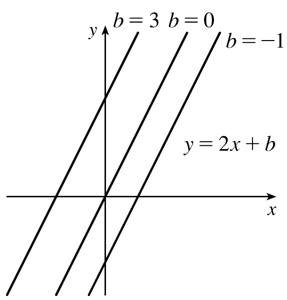
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

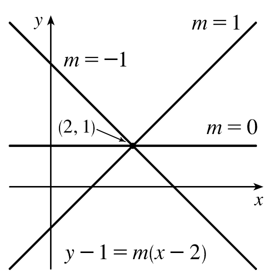
5. The denominator cannot equal 0, so $1 - \sin x \neq 0 \iff \sin x \neq 1 \iff x \neq \frac{\pi}{2} + 2n\pi$. Thus, the domain of $f(x) = \dfrac{\cos x}{1 - \sin x}$ is $\left\{x \mid x \neq \frac{\pi}{2} + 2n\pi,\ n \text{ an integer}\right\}$.

6. The denominator cannot equal 0, so $1 - \tan x \neq 0 \iff \tan x \neq 1 \iff x \neq \frac{\pi}{4} + n\pi$. The tangent function is not defined if $x \neq \frac{\pi}{2} + n\pi$. Thus, the domain of $g(x) = \dfrac{1}{1 - \tan x}$ is $\left\{x \mid x \neq \frac{\pi}{4} + n\pi,\ x \neq \frac{\pi}{2} + n\pi,\ n \text{ an integer}\right\}$.

7. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

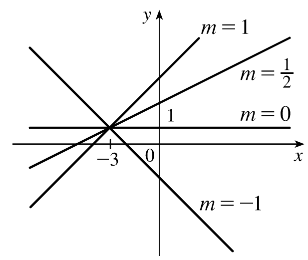


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

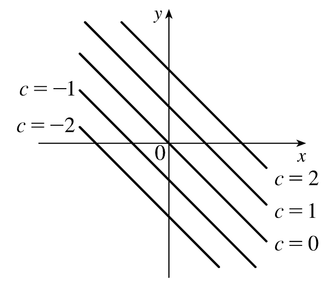


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

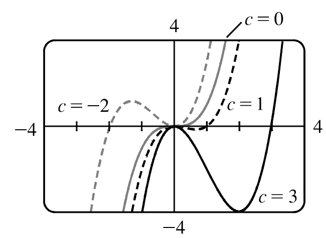
8. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



9. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



10. We graph $P(x) = x^3 - cx^2$ for $c = -2, 0, 1$, and 3. For $c \neq 0$, $P(x) = x^3 - cx^2 = x^2(x - c)$ has two x-intercepts, 0 and c. The curve has one decreasing portion that begins or ends at the origin and increases in length as $|c|$ increases; the decreasing portion is in quadrant II for $c < 0$ and in quadrant IV for $c > 0$.



11. Because f is a quadratic function, we know it is of the form $f(x) = ax^2 + bx + c$. The y-intercept is 18, so $f(0) = 18 \Rightarrow c = 18$ and $f(x) = ax^2 + bx + 18$. Since the points $(3, 0)$ and $(4, 2)$ lie on the graph of f, we have

$$f(3) = 0 \quad \Rightarrow \quad 9a + 3b + 18 = 0 \quad \Rightarrow \quad 3a + b = -6 \qquad \textbf{(1)}$$
$$f(4) = 2 \quad \Rightarrow \quad 16a + 4b + 18 = 2 \quad \Rightarrow \quad 4a + b = -4 \qquad \textbf{(2)}$$

This is a system of two equations in the unknowns a and b, and subtracting **(1)** from **(2)** gives $a = 2$. From **(1)**, $3(2) + b = -6 \Rightarrow b = -12$, so a formula for f is $f(x) = 2x^2 - 12x + 18$.

12. g is a quadratic function so $g(x) = ax^2 + bx + c$. The y-intercept is 1, so $g(0) = 1 \Rightarrow c = 1$ and $g(x) = ax^2 + bx + 1$. Since the points $(-2, 2)$ and $(1, -2.5)$ lie on the graph of g, we have

$$g(-2) = 2 \quad \Rightarrow \quad 4a - 2b + 1 = 2 \quad \Rightarrow \quad 4a - 2b = 1 \qquad \textbf{(1)}$$
$$g(1) = -2.5 \quad \Rightarrow \quad a + b + 1 = -2.5 \quad \Rightarrow \quad a + b = -3.5 \qquad \textbf{(2)}$$

Then **(1)** $+ 2 \cdot$ **(2)** gives us $6a = -6 \Rightarrow a = -1$ and from **(2)**, we have $-1 + b = -3.5 \Rightarrow b = -2.5$, so a formula for g is $g(x) = -x^2 - 2.5x + 1$.

13. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x-(-1)](x-0)(x-2)$, or $f(x) = ax(x+1)(x-2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.

$6 = a(1)(2)(-1) \quad \Rightarrow \quad -2a = 6 \quad \Rightarrow \quad a = -3$, so an equation for f is $f(x) = -3x(x+1)(x-2)$.

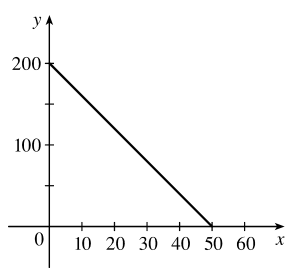
14. (a) For $T = 0.02t + 8.50$, the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of $0.02\ ^\circ\text{C}$ per year. The T-intercept is 8.50, which represents the average surface temperature in $^\circ\text{C}$ in the year 1900.

(b) $t = 2100 - 1900 = 200 \quad \Rightarrow \quad T = 0.02(200) + 8.50 = 12.50\,^\circ\text{C}$

15. (a) $D = 200$, so $c = 0.0417D(a+1) = 0.0417(200)(a+1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

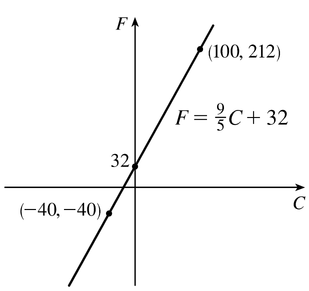
(b) For a newborn, $a = 0$, so $c = 8.34$ mg.

16. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

17. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1°C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

18. (a) Jari is traveling faster since the line representing her distance versus time is steeper than the corresponding line for Jade.

(b) At $t = 0$, Jade has traveled 10 miles. At $t = 6$, Jade has traveled 16 miles. Thus, Jade's speed is

$$\frac{16 \text{ miles} - 10 \text{ miles}}{6 \text{ minutes} - 0 \text{ minutes}} = 1 \text{ mi/min. This is } \frac{1 \text{ mile}}{1 \text{ minute}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = 60 \text{ mi/h}$$

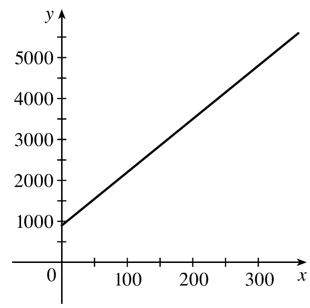
At $t = 0$, Jari has traveled 0 miles. At $t = 6$, Jari has traveled 7 miles. Thus, Jari's speed is

$$\frac{7 \text{ miles} - 0 \text{ miles}}{6 \text{ minutes} - 0 \text{ minutes}} = \frac{7}{6} \text{ mi/min or } \frac{7 \text{ miles}}{6 \text{ minutes}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = 70 \text{ mi/h}$$

(c) From part (b), we have a slope of 1 (mile/minute) for the linear function f modeling the distance traveled by Jade and from the graph the y-intercept is 10. Thus, $f(t) = 1t + 10 = t + 10$. Similarly, we have a slope of $\frac{7}{6}$ miles/minute for

Jari and a y-intercept of 0. Thus, the distance traveled by Jari as a function of time t (in minutes) is modeled by $g(t) = \frac{7}{6}t + 0 = \frac{7}{6}t$.

19. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$, we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \Leftrightarrow y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

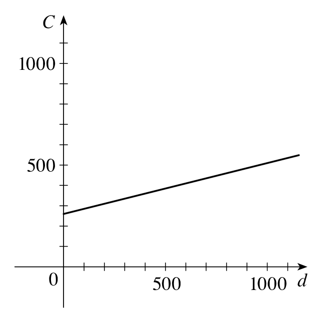
20. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$.

So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \Leftrightarrow C - 460 = \frac{1}{4}d - 200 \Leftrightarrow C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The C-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

21. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \Leftrightarrow 0.434d = 85 \Leftrightarrow d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is $100\ \text{lb/in}^2$ at a depth of approximately 196 feet.

22. (a) $R(x) = kx^{-2}$ and $R\,(0.005) = 140$, so $140 = k(0.005)^{-2} \Leftrightarrow k = 140(0.005)^2 = 0.0035$.

(b) $R(x) = 0.0035x^{-2}$, so for a diameter of 0.008 m the resistance is $R(0.008) = 0.0035(0.008)^{-2} \approx 54.7$ ohms.

23. If x is the original distance from the source, then the illumination is $f(x) = kx^{-2} = k/x^2$. Moving halfway to the lamp gives an illumination of $f\left(\frac{1}{2}x\right) = k\left(\frac{1}{2}x\right)^{-2} = k(2/x)^2 = 4(k/x^2)$, so the light is four times as bright.

24. (a) $P = k/V$ and $P = 39$ kPa when $V = 0.671\text{ m}^3$, so $39 = k/0.671 \quad \Leftrightarrow \quad k = 39(0.671) = 26.169$.

(b) When $V = 0.916$, $P = 26.169/V = 26.169/0.916 \approx 28.6$, so the pressure is reduced to approximately 28.6 kPa.

25. (a) $P = kAv^3$ so doubling the windspeed v gives $P = kA(2v)^3 = 8(kAv^3)$. Thus, the power output is increased by a factor of eight.

(b) The area swept out by the blades is given by $A = \pi l^2$, where l is the blade length, so the power output is $P = kAv^3 = k\pi l^2 v^3$. Doubling the blade length gives $P = k\pi(2l)^2 v^3 = 4(k\pi l^2 v^3)$. Thus, the power output is increased by a factor of four.

(c) From part (b) we have $P = k\pi l^2 v^3$, and $k = 0.214\text{ kg/m}^3$, $l = 30$ m gives

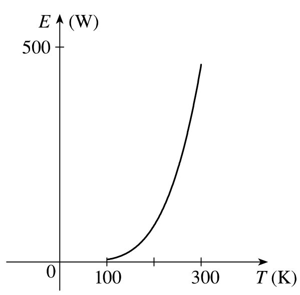
$$P = 0.214\,\frac{\text{kg}}{\text{m}^3}\cdot 900\pi\text{ m}^2\cdot v^3 = 192.6\pi v^3\,\frac{\text{kg}}{\text{m}}$$

For $v = 10$ m/s, we have

$$P = 192.6\pi\left(10\,\frac{\text{m}}{\text{s}}\right)^3\frac{\text{kg}}{\text{m}} = 192{,}600\pi\,\frac{\text{m}^2\cdot\text{kg}}{\text{s}^3} \approx 605{,}000\text{ W}$$

Similarly, $v = 15$ m/s gives $P = 650{,}025\pi \approx 2{,}042{,}000$ W and $v = 25$ m/s gives $P = 3{,}009{,}375\pi \approx 9{,}454{,}000$ W.

26. (a) We graph $E(T) = (5.67\times 10^{-8})T^4$ for $100 \le T \le 300$:



(b) From the graph, we see that as temperature increases, energy increases—slowly at first, but then at an increasing rate.

27. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

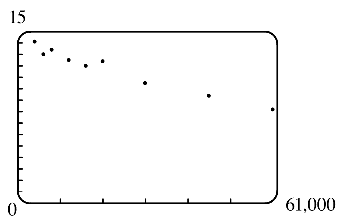
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

28. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a\cdot b^x$ or $f(x) = a\cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Exercises 29–33: Some values are given to many decimal places. The results may depend on the technology used—rounding is left to the reader.

29. (a)

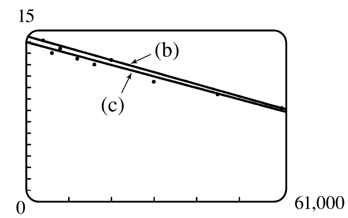


A linear model does seem appropriate.

(b) Using the points (4000, 14.1) and (60,000, 8.2), we obtain

$y - 14.1 = \dfrac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000)$ or, equivalently,

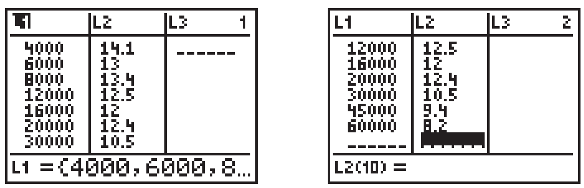
$y \approx -0.000105357x + 14.521429$.



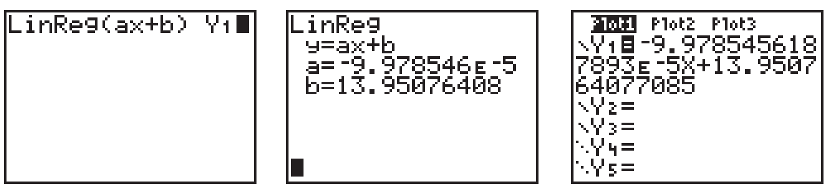
(c) Using a computing device, we obtain the regression line $y = -0.0000997855x + 13.950764$.

The following commands and screens illustrate how to find the regression line on a TI-84 Plus calculator.

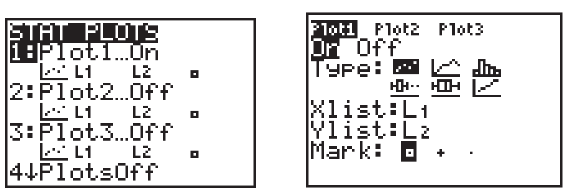
Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.



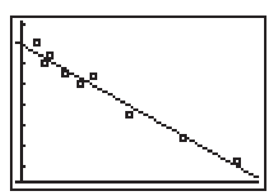
Find the regession line and store it in Y_1. Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER .



Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER .



Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.



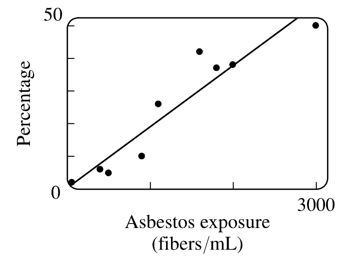
(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

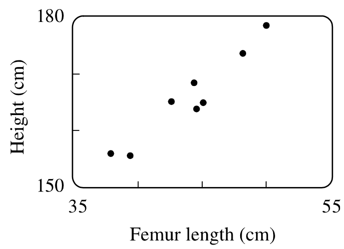
30. (a) Using a computing device, we obtain the regression line $y = 0.01879x + 0.30480$.

(b) The regression line appears to be a suitable model for the data.

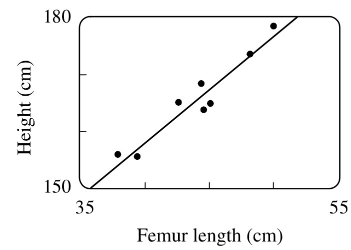


(c) The y-intercept represents the percentage of laboratory rats that develop lung tumors when *not* exposed to asbestos fibers.

31. (a)



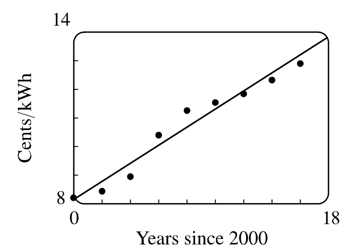
(b) Using a computing device, we obtain the regression line $y = 1.88074x + 82.64974$.



(c) When $x = 53$ cm, $y \approx 182.3$ cm.

32. (a) See the scatter plot in part (b). A linear model seems appropriate.

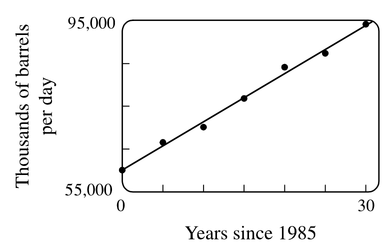
(b) Using a computing device, we obtain the regression line $y = 0.31567x + 8.15578$.



(c) For 2005, $x = 5$ and $y \approx 9.73$ cents/kWh. For 2017, $x = 17$ and $y \approx 13.52$ cents/kWh.

33. (a) See the scatter plot in part (b). A linear model seems appropriate.

(b) Using a computing device, we obtain the regression line $y = 1124.86x + 60{,}119.86$.



(c) For 2002, $x = 17$ and $y \approx 79{,}242$ thousands of barrels per day. For 2017, $x = 32$ and $y \approx 96{,}115$ thousands of barrels per day.

34. (a) $T = 1.000\,431\,227 d^{\,1.499\,528\,750}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^{\,3}$.

35. (a) If $A = 60$, then $S = 0.7A^{0.3} \approx 2.39$, so you would expect to find 2 species of bats in that cave.

(b) $S = 4 \quad \Rightarrow \quad 4 = 0.7A^{0.3} \quad \Rightarrow \quad \frac{40}{7} = A^{3/10} \quad \Rightarrow \quad A = \left(\frac{40}{7}\right)^{10/3} \approx 333.6$, so we estimate the surface area of the cave to be 334 m^2.

36. (a) Using a computing device, we obtain a power function $N = cA^b$, where $c \approx 3.1046$ and $b \approx 0.308$.

(b) If $A = 291$, then $N = cA^b \approx 17.8$, so you would expect to find 18 species of reptiles and amphibians on Dominica.

37. We have $I = \dfrac{S}{4\pi r^2} = \left(\dfrac{S}{4\pi}\right)\left(\dfrac{1}{r^2}\right) = \dfrac{S/(4\pi)}{r^2}$. Thus, $I = \dfrac{k}{r^2}$ with $k = \dfrac{S}{4\pi}$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = f(x) + 8$ from the graph of $y = f(x)$, shift the graph 8 units upward.

(b) To obtain the graph of $y = f(x + 8)$ from the graph of $y = f(x)$, shift the graph 8 units to the left.

(c) To obtain the graph of $y = 8f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 8.

(d) To obtain the graph of $y = f(8x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of $y = -f(x) - 1$ from the graph of $y = f(x)$, first reflect the graph about the x-axis, and then shift it 1 unit downward.

(f) To obtain the graph of $y = 8f(\frac{1}{8}x)$ from the graph of $y = f(x)$, stretch the graph horizontally and vertically by a factor of 8.

3. (a) *Graph 3:* The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

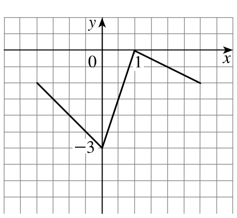
(b) *Graph 1:* The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) *Graph 4:* The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

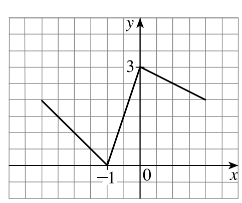
(d) *Graph 5:* The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

(e) *Graph 2:* The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

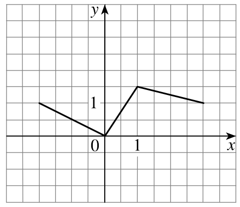
4. (a) $y = f(x) - 3$: Shift the graph of f 3 units down.



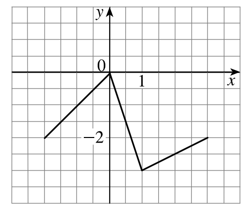
(b) $y = f(x + 1)$: Shift the graph of f 1 unit to the left.



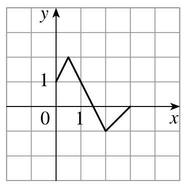
(c) $y = \frac{1}{2}f(x)$: Shrink the graph of f vertically by a factor of 2.



(d) $y = -f(x)$: Reflect the graph of f about the x-axis.

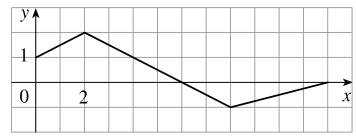


5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



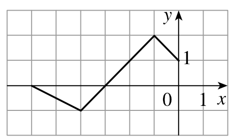
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



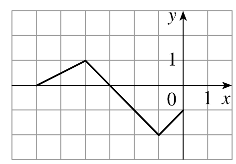
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$y = 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} = 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10}$$

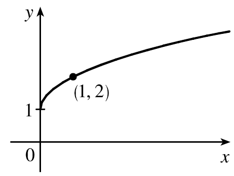
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about } x\text{-axis}}} \quad f \underbrace{(x+4)}_{\substack{\text{shift} \\ \text{4 units left}}} \quad \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

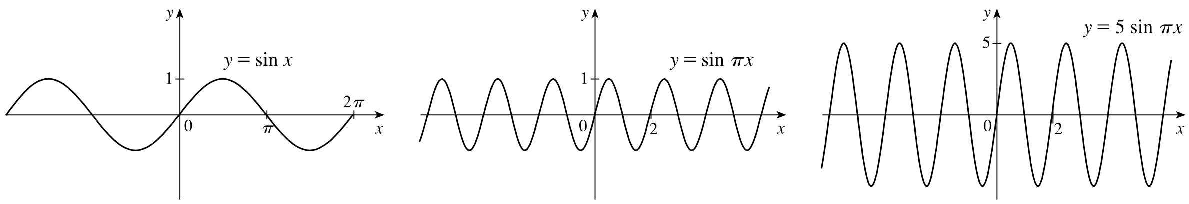
This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 \\ &= -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

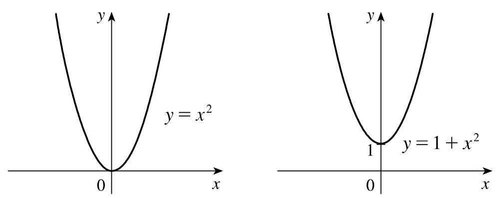
8. (a) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



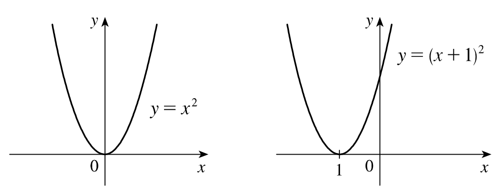
(b) The graph of $y = \sin \pi x$ can be obtained from the graph of $y = \sin x$ by compressing horizontally by a factor of π, giving a period of $2\pi/\pi = 2$. The graph of $y = 5 \sin \pi x$ is then obtained by stretching vertically by a factor of 5.



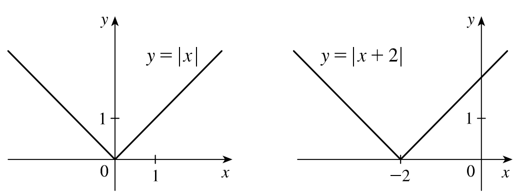
9. $y = 1 + x^2$. Start with the graph of $y = x^2$ and shift 1 unit upward



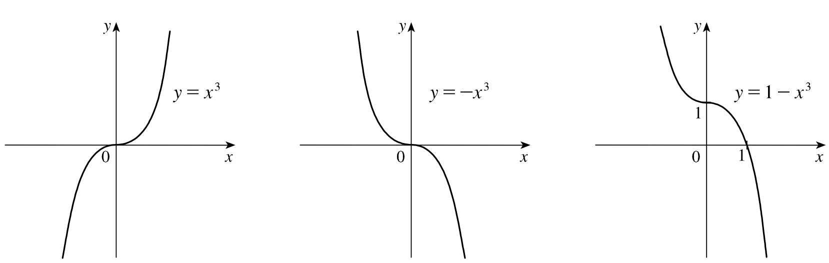
10. $y = (x + 1)^2$. Start with the graph of $y = x^2$ and shift 1 unit to the left.



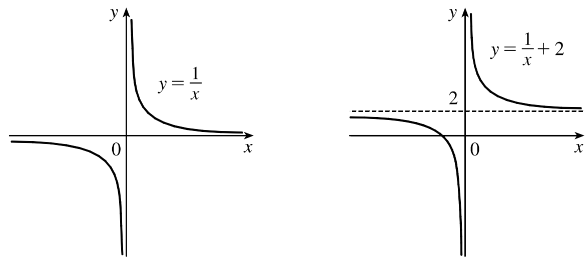
11. $y = |x + 2|$. Start with the graph of $y = |x|$ and shift 2 units to the left.



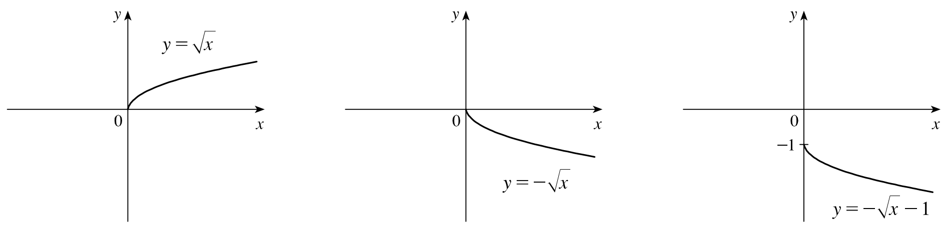
12. $y = 1 - x^3$. Start with the graph of $y = x^3$, reflect about the x-axis, and then shift 1 unit upward.



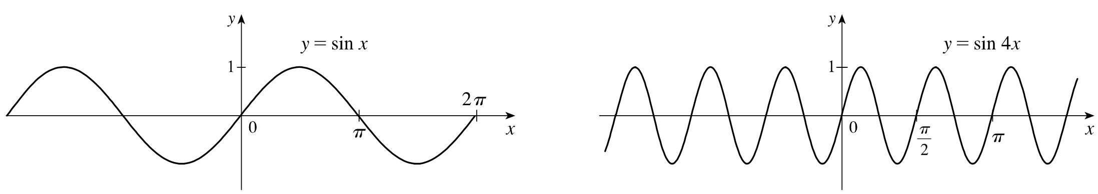
13. $y = \dfrac{1}{x} + 2$. Start with the graph of $y = \dfrac{1}{x}$ and shift 2 units upward.



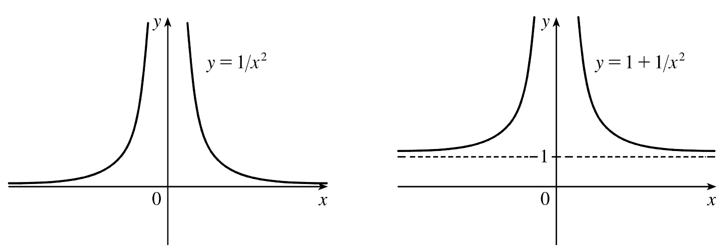
14. $y = -\sqrt{x} - 1$. Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and then shift 1 unit downward.



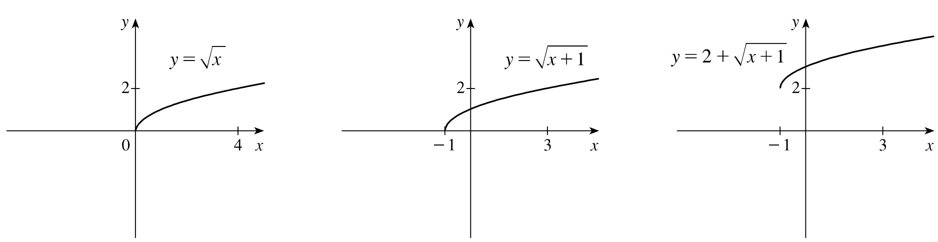
15. $y = \sin 4x$. Start with the graph of $y = \sin x$ and compress horizontally by a factor of 4. The period becomes $2\pi/4 = \pi/2$.



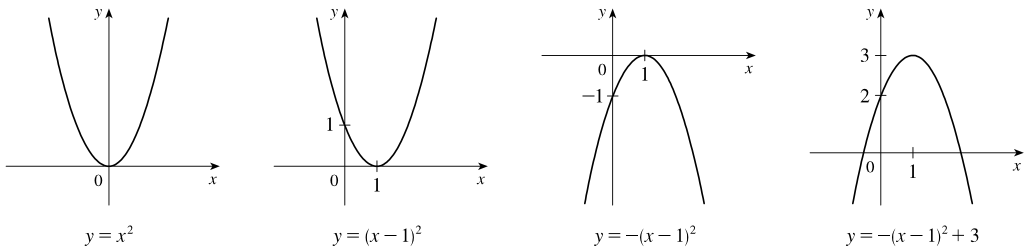
16. $y = 1 + \dfrac{1}{x^2}$. Start with the graph of $y = \dfrac{1}{x^2}$ and shift 1 unit upward.



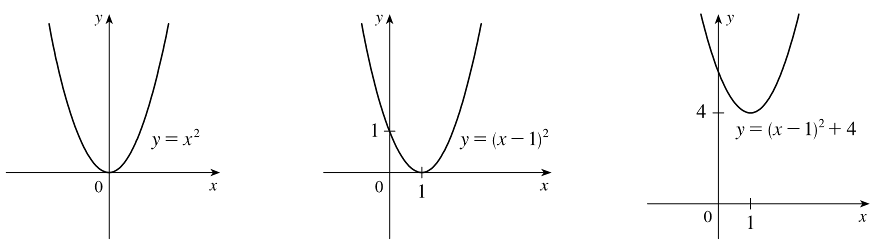
17. $y = 2 + \sqrt{x + 1}$. Start with the graph of $y = \sqrt{x}$, shift 1 unit to the left, and then shift 2 units upward.



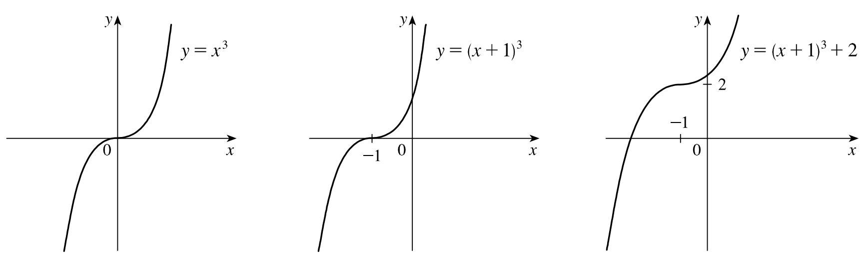
18. $y = -(x-1)^2 + 3$. Start with the graph of $y = x^2$, shift 1 unit to the right, reflect about the x-axis, and then shift 3 units upward.



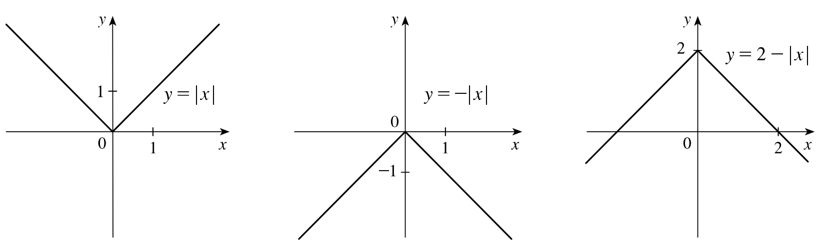
19. $y = x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x-1)^2 + 4$. Start with the graph of $y = x^2$, shift 1 unit to the right, and then shift 4 units upward.



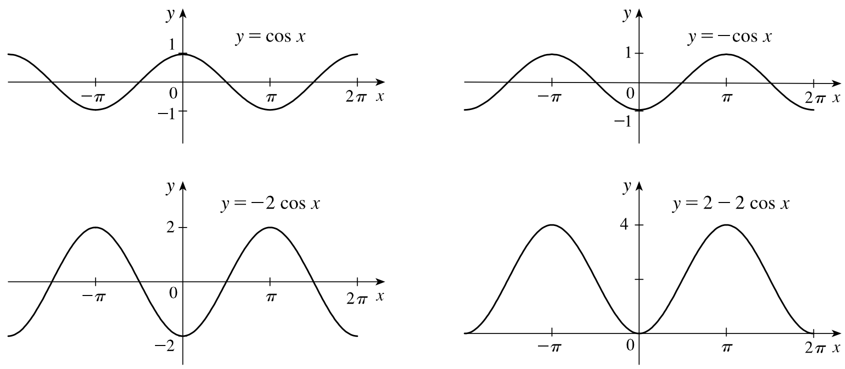
20. $y = (x+1)^3 + 2$. Start with the graph of $y = x^3$, shift 1 unit to the left, and then shift 2 units upward.



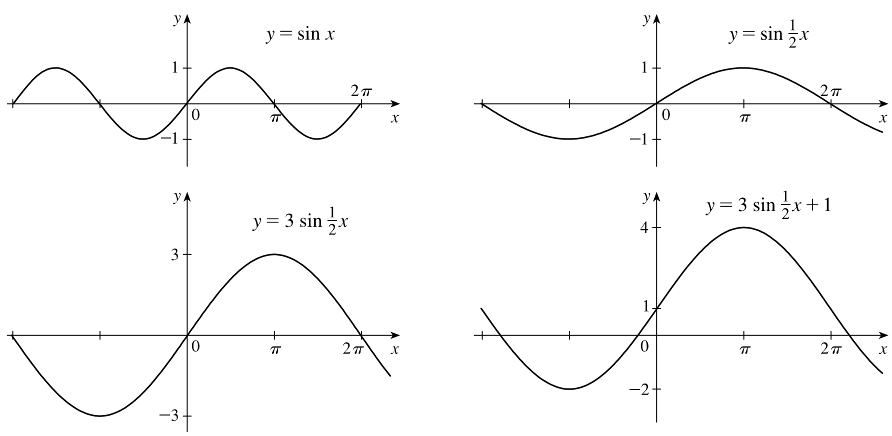
21. $y = 2 - |x|$. Start with the graph of $y = |x|$, reflect about the x-axis, and then shift 2 units upward.



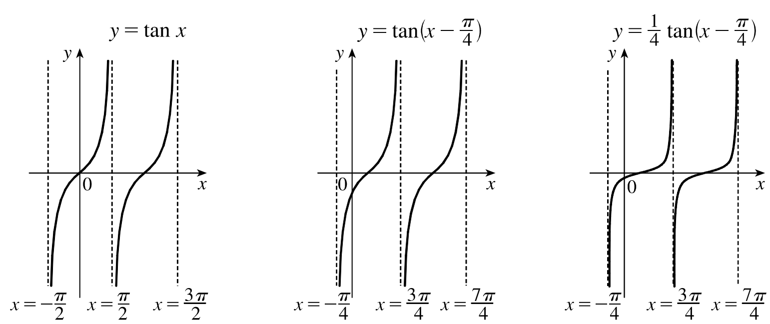
22. $y = 2 - 2\cos x$. Start with the graph of $y = \cos x$, reflect about the x-axis, stretch vertically by a factor of 2, and then shift 2 units upward.



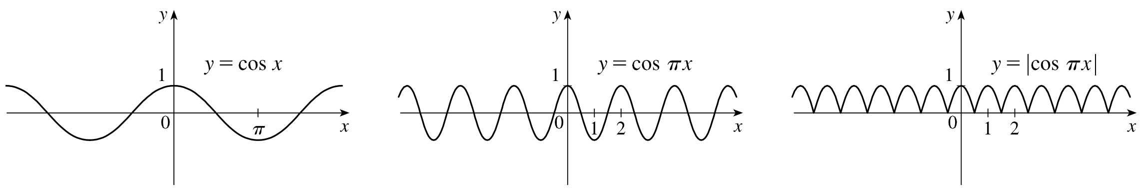
23. $y = 3\sin\frac{1}{2}x + 1$. Start with the graph of $y = \sin x$, stretch horizontally by a factor of 2, stretch vertically by a factor of 3, and then shift 1 unit upward.



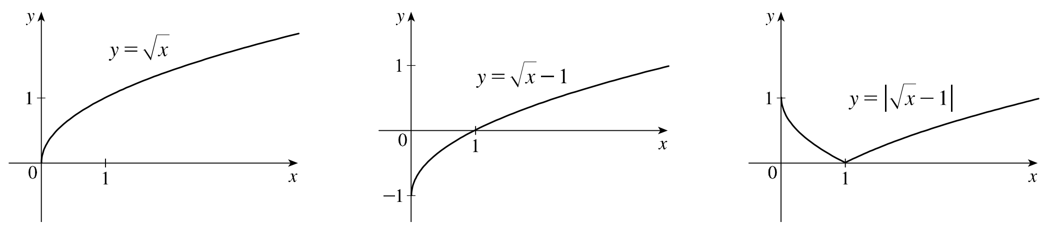
24. $y = \frac{1}{4}\tan\left(x - \frac{\pi}{4}\right)$. Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



25. $y = |\cos \pi x|$. Start with the graph of $y = \cos x$, shrink horizontally by a factor of π, and reflect all the parts of the graph below the x-axis about the x-axis.



26. $y = \left|\sqrt{x} - 1\right|$. Start with the graph of $y = \sqrt{x}$, shift 1 unit downward, and then reflect the portion of the graph below the x-axis about the x-axis.



27. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

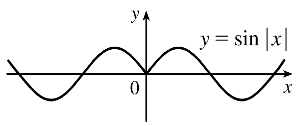
28. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

29. The water depth $D(t)$ can be modeled by a cosine function with amplitude $\dfrac{12 - 2}{2} = 5$ m, average magnitude $\dfrac{12 + 2}{2} = 7$ m, and period 12 hours. High tide occurred at time 6:45 AM ($t = 6.75$ h), so the curve begins a cycle at time $t = 6.75$ h (shift 6.75 units to the right). Thus, $D(t) = 5\cos\left[\frac{2\pi}{12}(t - 6.75)\right] + 7 = 5\cos\left[\frac{\pi}{6}(t - 6.75)\right] + 7$, where D is in meters and t is the number of hours after midnight.

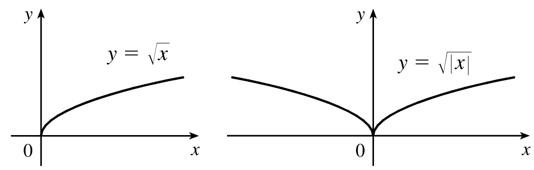
30. The total volume of air $V(t)$ in the lungs can be modeled by a sine function with amplitude $\dfrac{2500 - 2000}{2} = 250$ mL, average volume $\dfrac{2500 + 2000}{2} = 2250$ mL, and period 4 seconds. Thus, $V(t) = 250\sin\frac{2\pi}{4}t + 2250 = 250\sin\frac{\pi}{2}t + 2250$, where V is in mL and t is in seconds.

31. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

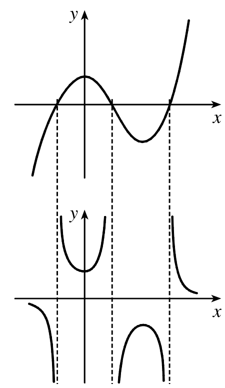
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$



32. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



33. $f(x) = \sqrt{25 - x^2}$ is defined only when $25 - x^2 \geq 0 \quad \Leftrightarrow \quad x^2 \leq 25 \quad \Leftrightarrow \quad -5 \leq x \leq 5$, so the domain of f is $[-5, 5]$. For $g(x) = \sqrt{x+1}$, we must have $x + 1 \geq 0 \quad \Leftrightarrow \quad x \geq -1$, so the domain of g is $[-1, \infty)$.

(a) $(f + g)(x) = \sqrt{25 - x^2} + \sqrt{x+1}$. The domain of $f + g$ is found by intersecting the domains of f and g: $[-1, 5]$.

(b) $(f - g)(x) = \sqrt{25 - x^2} - \sqrt{x+1}$. The domain of $f - g$ is found by intersecting the domains of f and g: $[-1, 5]$.

(c) $(fg)(x) = \sqrt{25 - x^2} \cdot \sqrt{x+1} = \sqrt{-x^3 - x^2 + 25x + 25}$. The domain of fg is found by intersecting the domains of f and g: $[-1, 5]$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{25 - x^2}}{\sqrt{x+1}} = \sqrt{\dfrac{25 - x^2}{x+1}}$. Notice that we must have $x + 1 \neq 0$ in addition to any previous restrictions. Thus, the domain of f/g is $(-1, 5]$.

34. For $f(x) = \dfrac{1}{x-1}$, we must have $x - 1 \neq 0 \quad \Leftrightarrow \quad x \neq 1$. For $g(x) = \dfrac{1}{x} - 2$, we must have $x \neq 0$.

(a) $(f + g)(x) = \dfrac{1}{x-1} + \dfrac{1}{x} - 2 = \dfrac{x + x - 1 - 2x(x-1)}{x(x-1)} = \dfrac{2x - 1 - 2x^2 + 2x}{x^2 - x} = -\dfrac{2x^2 - 4x + 1}{x^2 - x}, \quad \{x \mid x \neq 0, 1\}$

(b) $(f - g)(x) = \dfrac{1}{x-1} - \left(\dfrac{1}{x} - 2\right) = \dfrac{x - (x-1) + 2x(x-1)}{x(x-1)} = \dfrac{1 + 2x^2 - 2x}{x^2 - x} = \dfrac{2x^2 - 2x + 1}{x^2 - x}, \quad \{x \mid x \neq 0, 1\}$

(c) $(fg)(x) = \dfrac{1}{x-1}\left(\dfrac{1}{x} - 2\right) = \dfrac{1}{x^2 - x} - \dfrac{2}{x-1} = \dfrac{1 - 2x}{x^2 - x}, \quad \{x \mid x \neq 0, 1\}$

(d) $$\left(\frac{f}{g}\right)(x) = \frac{\dfrac{1}{x-1}}{\dfrac{1}{x} - 2} = \frac{\dfrac{1}{x-1}}{\dfrac{1-2x}{x}} = \frac{1}{x-1} \cdot \frac{x}{1-2x} = \frac{x}{(x-1)(1-2x)} = -\frac{x}{(x-1)(2x-1)}$$
$$= -\frac{x}{2x^2 - 3x + 1}, \quad \{x \mid x \neq 0, \tfrac{1}{2}, 1\}$$

[Note the additional domain restriction $g(x) \neq 0 \quad \Rightarrow \quad x \neq \frac{1}{2}$.]

35. $f(x) = x^3 + 5$ and $g(x) = \sqrt[3]{x}$. The domain of each function is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 + 5 = x + 5$. The domain is $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^3 + 5) = \sqrt[3]{x^3 + 5}$. The domain is $(-\infty, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^3 + 5) = (x^3 + 5)^3 + 5$. The domain is $(-\infty, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{x}\right) = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x}$. The domain is $(-\infty, \infty)$.

36. $f(x) = 1/x$ and $g(x) = 2x + 1$. The domain of f is $(-\infty, 0) \cup (0, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x + 1) = \dfrac{1}{2x + 1}$. The domain is $\{x \mid 2x + 1 \neq 0\} = \{x \mid x \neq -\frac{1}{2}\} = \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x}\right) = 2\left(\dfrac{1}{x}\right) + 1 = \dfrac{2}{x} + 1$. We must have $x \neq 0$, so the domain is $(-\infty, 0) \cup (0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{1/x} = x$. Since f requires $x \neq 0$, the domain is $(-\infty, 0) \cup (0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x + 1) = 2(2x + 1) + 1 = 4x + 3$. The domain is $(-\infty, \infty)$.

37. $f(x) = \dfrac{1}{\sqrt{x}}$ and $g(x) = x + 1$. The domain of f is $(0, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(x + 1) = \dfrac{1}{\sqrt{x + 1}}$. We must have $x + 1 > 0$, or $x > -1$, so the domain is $(-1, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{\sqrt{x}} + 1$. We must have $x > 0$, so the domain is $(0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{1}{\sqrt{1/\sqrt{x}}} = \dfrac{1}{1/\sqrt{\sqrt{x}}} = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. We must have $x > 0$, so the domain is $(0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(x + 1) = (x + 1) + 1 = x + 2$. The domain is $(-\infty, \infty)$.

38. $f(x) = \dfrac{x}{x + 1}$ and $g(x) = 2x - 1$. The domain of f is $(-\infty, -1) \cup (-1, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x - 1) = \dfrac{2x - 1}{(2x - 1) + 1} = \dfrac{2x - 1}{2x}$. We must have $2x \neq 0 \iff x \neq 0$. Thus, the domain is $(-\infty, 0) \cup (0, \infty)$.

(b) $(g \circ f)(x) = g(f(x)) = 2\left(\dfrac{x}{x + 1}\right) - 1 = \dfrac{2x}{x + 1} - 1 = \dfrac{2x - 1(x + 1)}{x + 1} = \dfrac{x - 1}{x + 1}$. We must have $x + 1 \neq 0 \iff x \neq -1$. Thus, the domain is $(-\infty, -1) \cup (-1, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = \dfrac{\frac{x}{x + 1}}{\frac{x}{x + 1} + 1} = \dfrac{\frac{x}{x + 1}}{\frac{x}{x + 1} + 1} \cdot \dfrac{x + 1}{x + 1} = \dfrac{x}{x + (x + 1)} = \dfrac{x}{2x + 1}$. We must have both $x + 1 \neq 0$ and $2x + 1 \neq 0$, so the domain excludes both -1 and $-\frac{1}{2}$. Thus, the domain is $(-\infty, -1) \cup \left(-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x - 1) = 2(2x - 1) - 1 = 4x - 3$. The domain is $(-\infty, \infty)$.

39. $f(x) = \dfrac{2}{x}$ and $g(x) = \sin x$. The domain of f is $(-\infty, 0) \cup (0, \infty)$. The domain of g is $(-\infty, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f(\sin x) = \dfrac{2}{\sin x} = 2\csc x$. We must have $\sin x \neq 0$, so the domain is $\{x \mid x \neq k\pi, k \text{ an integer}\}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{2}{x}\right) = \sin\left(\dfrac{2}{x}\right)$. We must have $x \neq 0$, so the domain is $(-\infty, 0) \cup (0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{2}{x}\right) = \dfrac{2}{\frac{2}{x}} = x$. Since f requires $x \neq 0$, the domain is $(-\infty, 0) \cup (0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(\sin x) = \sin(\sin x)$. The domain is $(-\infty, \infty)$.

40. $f(x) = \sqrt{5-x}$ and $g(x) = \sqrt{x-1}$. The domain of f is $(-\infty, 5]$ and the domain of g is $[1, \infty)$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x-1}\right) = \sqrt{5 - \sqrt{x-1}}$. We must have $x - 1 \geq 0 \Leftrightarrow x \geq 1$ and $5 - \sqrt{x-1} \geq 0 \Leftrightarrow \sqrt{x-1} \leq 5 \Leftrightarrow 0 \leq x - 1 \leq 25 \Leftrightarrow 1 \leq x \leq 26$. Thus, the domain is $[1, 26]$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(\sqrt{5-x}\right) = \sqrt{\sqrt{5-x} - 1}$. We must have $5 - x \geq 0 \Leftrightarrow x \leq 5$ and $\sqrt{5-x} - 1 \geq 0 \Leftrightarrow \sqrt{5-x} \geq 1 \Leftrightarrow 5 - x \geq 1 \Leftrightarrow x \leq 4$. Intersecting the restrictions on x gives a domain of $(-\infty, 4]$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\sqrt{5-x}\right) = \sqrt{5 - \sqrt{5-x}}$. We must have $5 - x \geq 0 \Leftrightarrow x \leq 5$ and $5 - \sqrt{5-x} \geq 0 \Leftrightarrow \sqrt{5-x} \leq 5 \Leftrightarrow 0 \leq 5 - x \leq 25 \Leftrightarrow -5 \leq -x \leq 20 \Leftrightarrow -20 \leq x \leq 5$. Intersecting the restrictions on x gives a domain of $[-20, 5]$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt{x-1}\right) = \sqrt{\sqrt{x-1} - 1}$. We must have $x - 1 \geq 0 \Leftrightarrow x \geq 1$ and $\sqrt{x-1} - 1 \geq 0 \Leftrightarrow \sqrt{x-1} \geq 1 \Leftrightarrow x - 1 \geq 1 \Leftrightarrow x \geq 2$. Intersecting the restrictons on x gives a domain of $[2, \infty)$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(\sin(x^2)) = 3\sin(x^2) - 2$

42. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt{x}\right)\right) = f\left(2^{\sqrt{x}}\right) = \left|2^{\sqrt{x}} - 4\right|$

43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3 + 2)) = f[(x^3 + 2)^2] = f(x^6 + 4x^3 + 4)$

$$= \sqrt{(x^6 + 4x^3 + 4) - 3} = \sqrt{x^6 + 4x^3 + 1}$$

44. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt[3]{x}\right)\right) = f\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right) = \tan\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right)$

45. Let $g(x) = 2x + x^2$ and $f(x) = x^4$. Then $(f \circ g)(x) = f(g(x)) = f(2x + x^2) = (2x + x^2)^4 = F(x)$.

46. Let $g(x) = \cos x$ and $f(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(\cos x) = (\cos x)^2 = \cos^2 x = F(x)$.

47. Let $g(x) = \sqrt[3]{x}$ and $f(x) = \dfrac{x}{1+x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x}\right) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}} = F(x)$.

48. Let $g(x) = \dfrac{x}{1+x}$ and $f(x) = \sqrt[3]{x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{1+x}\right) = \sqrt[3]{\dfrac{x}{1+x}} = G(x)$.

49. Let $g(t) = t^2$ and $f(t) = \sec t \, \tan t$. Then $(f \circ g)(t) = f(g(t)) = f(t^2) = \sec(t^2)\,\tan(t^2) = v(t)$.

50. Let $g(x) = \sqrt{x}$ and $f(x) = \sqrt{1+x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \sqrt{1+\sqrt{x}} = H(x)$.

51. Let $h(x) = \sqrt{x}$, $g(x) = x - 1$, and $f(x) = \sqrt{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt{x}\right)\right) = f\left(\sqrt{x} - 1\right) = \sqrt{\sqrt{x} - 1} = R(x)$.

52. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[8]{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2 + |x|) = \sqrt[8]{2 + |x|} = H(x)$.

53. Let $h(t) = \cos t$, $g(t) = \sin t$, and $f(t) = t^2$. Then

$(f \circ g \circ h)(t) = f(g(h(t))) = f(g(\cos t)) = f(\sin(\cos t)) = [\sin(\cos t)]^2 = \sin^2(\cos t) = S(t)$.

54. Let $h(t) = \tan t$, $g(t) = \sqrt{t} + 1$, and $f(t) = \cos t$. Then

$(f \circ g \circ h)(t) = f(g(h(t))) = f(g(\tan t)) = f\left(\sqrt{\tan t} + 1\right) = \cos\left(\sqrt{\tan t} + 1\right) = H(t)$.

55. (a) $f(g(3)) = f(4) = 6$.
(b) $g(f(2)) = g(1) = 5$.
(c) $(f \circ g)(5) = f(g(5)) = f(3) = 5$.
(d) $(g \circ f)(5) = g(f(5)) = g(2) = 3$.

56. (a) $g(g(g(2))) = g(g(3)) = g(4) = 1$.
(b) $(f \circ f \circ f)(1) = f(f(f(1))) = f(f(3)) = f(5) = 2$.
(c) $(f \circ f \circ g)(1) = f(f(g(1))) = f(f(5)) = f(2) = 1$.
(d) $(g \circ f \circ g)(3) = g(f(g(3))) = g(f(4)) = g(6) = 2$.

57. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

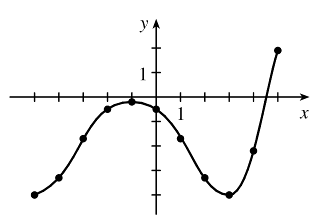
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

58. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



59. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

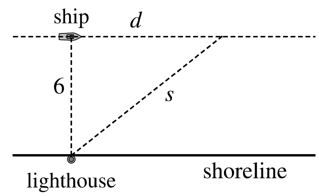
(b) $A = \pi r^2 \quad \Rightarrow \quad (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

60. (a) The radius r of the balloon is increasing at a rate of 2 cm/s, so $r(t) = (2\text{ cm/s})(t\text{ s}) = 2t$ (in cm).

(b) Using $V = \frac{4}{3}\pi r^3$, we get $(V \circ r)(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3$.

The result, $V = \frac{32}{3}\pi t^3$, gives the volume of the balloon (in cm^3) as a function of time (in s).

61. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s. By the Pythagorean Theorem, $d^2 + 6^2 = s^2 \quad \Rightarrow \quad s = f(d) = \sqrt{d^2 + 36}$.



(b) Using $d = rt$, we get $d = (30\text{ km/h})(t\text{ hours}) = 30t$ (in km). Thus, $d = g(t) = 30t$.

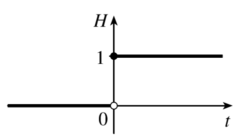
(c) $(f \circ g)(t) = f(g(t)) = f(30t) = \sqrt{(30t)^2 + 36} = \sqrt{900t^2 + 36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

62. (a) $d = rt \quad \Rightarrow \quad d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

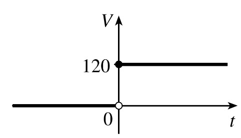
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

63. (a)



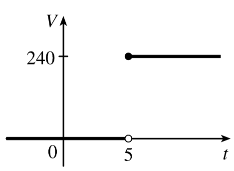
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

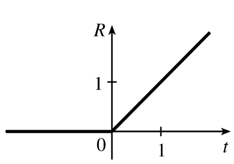
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

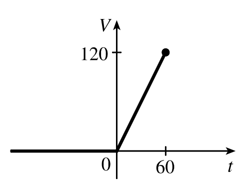
64. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



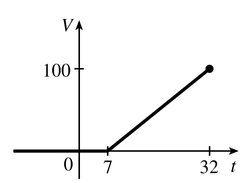
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t),\ t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t-7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t - 7)H(t - 7),\ t \leq 32$.



65. If $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2x + b_2) = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1.$

So $f \circ g$ is a linear function with slope m_1m_2.

66. If $A(x) = 1.04x$, then

$(A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2x,$

$(A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2x) = 1.04(1.04)^2x = (1.04)^3x$, and

$(A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3x) = 1.04(1.04)^3x, = (1.04)^4x.$

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose n copies of A, we get the formula $\underbrace{(A \circ A \circ \cdots \circ A)}_{n\ A\text{'s}}(x) = (1.04)^nx$.

67. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

68. We need a function g so that $g(f(x)) = g(x + 4) = h(x) = 4x - 1 = 4(x + 4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

69. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

70. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

71. (a) $E(x) = f(x) + f(-x) \quad \Rightarrow \quad E(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = E(x)$. Since $E(-x) = E(x)$, E is an even function.

(b) $O(x) = f(x) - f(-x) \quad \Rightarrow \quad O(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -[f(x) - f(-x)] = -O(x)$.

Since $O(-x) = -O(x)$, O is an odd function.

(c) For any function f with domain $\mathbb{R}$, define functions E and O as in parts (a) and (b). Then $\frac{1}{2}E$ is even, $\frac{1}{2}O$ is odd, and we show that $f(x) = \frac{1}{2}E(x) + \frac{1}{2}O(x)$:

$$\begin{aligned}\tfrac{1}{2}E(x) + \tfrac{1}{2}O(x) &= \tfrac{1}{2}\left[f(x) + f(-x)\right] + \tfrac{1}{2}\left[f(x) - f(-x)\right]\\ &= \tfrac{1}{2}[f(x) + f(-x) + f(x) - f(-x)]\\ &= \tfrac{1}{2}[2f(x)] = f(x)\end{aligned}$$

as desired.

(d) $f(x) = 2^x + (x-3)^2$ has domain $\mathbb{R}$, so we know from part (c) that $f(x) = \frac{1}{2}E\,(x) + \frac{1}{2}O\,(x)$, where

$$\begin{aligned}E(x) = f(x) + f\,(-x) &= 2^x + (x-3)^2 + 2^{-x} + (-x-3)^2\\ &= 2^x + 2^{-x} + (x-3)^2 + (x+3)^2\end{aligned}$$

and

$$\begin{aligned}O(x) = f(x) - f\,(-x) &= 2^x + (x-3)^2 - [2^{-x} + (-x-3)^2]\\ &= 2^x - 2^{-x} + (x-3)^2 - (x+3)^2\end{aligned}$$

1.4 The Tangent and Velocity Problems

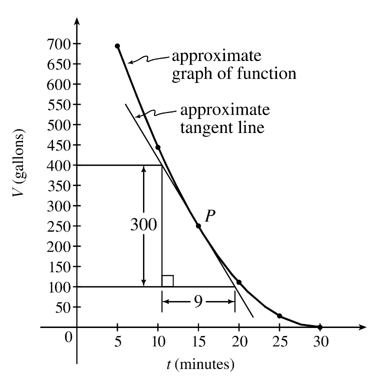
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) (i) On the interval $[0, 40]$, slope $= \dfrac{7398 - 3438}{40 - 0} = 99$.

(ii) On the interval $[10, 20]$, slope $= \dfrac{5622 - 4559}{20 - 10} = 106.3$.

(iii) On the interval $[20, 30]$, slope $= \dfrac{6536 - 5622}{30 - 20} = 91.4$.

The slopes represent the average number of steps per minute the student walked during the respective time intervals.

(b) Averaging the slopes of the secant lines corresponding to the intervals immediately before and after $t = 20$, we have

$$\frac{106.3 + 91.4}{2} = 98.85$$

The student's walking pace is approximately 99 steps per minute at 3:20 PM.

3. (a) $y = \dfrac{1}{1 - x}$, $P(2, -1)$

	x	$Q(x, 1/(1-x))$	m_{PQ}
(i)	1.5	$(1.5, -2)$	2
(ii)	1.9	$(1.9, -1.111\,111)$	1.111 111
(iii)	1.99	$(1.99, -1.010\,101)$	1.010 101
(iv)	1.999	$(1.999, -1.001\,001)$	1.001 001
(v)	2.5	$(2.5, -0.666\,667)$	0.666 667
(vi)	2.1	$(2.1, -0.909\,091)$	0.909 091
(vii)	2.01	$(2.01, -0.990\,099)$	0.990 099
(viii)	2.001	$(2.001, -0.999\,001)$	0.999 001

(b) The slope appears to be 1.

(c) Using $m = 1$, an equation of the tangent line to the curve at $P(2, -1)$ is $y - (-1) = 1(x - 2)$, or $y = x - 3$.

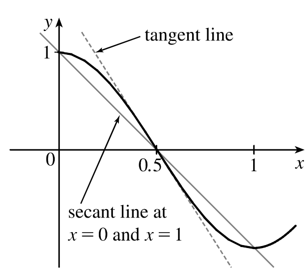
4. (a) $y = \cos \pi x$, $P(0.5, 0)$

	x	Q	m_{PQ}
(i)	0	$(0, 1)$	-2
(ii)	0.4	$(0.4, 0.309017)$	-3.090170
(iii)	0.49	$(0.49, 0.031411)$	-3.141076
(iv)	0.499	$(0.499, 0.003142)$	-3.141587
(v)	1	$(1, -1)$	-2
(vi)	0.6	$(0.6, -0.309017)$	-3.090170
(vii)	0.51	$(0.51, -0.031411)$	-3.141076
(viii)	0.501	$(0.501, -0.003142)$	-3.141587

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)



5. (a) $y = y(t) = 275 - 16t^2$. At $t = 4$, $y = 275 - 16(4)^2 = 19$. The average velocity between times 4 and $4 + h$ is

$$v_{\text{avg}} = \frac{y(4 + h) - y(4)}{(4 + h) - 4} = \frac{\left[275 - 16(4 + h)^2\right] - 19}{h} = \frac{-128h - 16h^2}{h} = -128 - 16h \quad \text{if } h \neq 0$$

(i) 0.1 seconds: $h = 0.1$, $v_{\text{avg}} = -129.6$ ft/s

(ii) 0.05 seconds: $h = 0.05$, $v_{\text{avg}} = -128.8$ ft/s

(iii) 0.01 seconds: $h = 0.01$, $v_{\text{avg}} = -128.16$ ft/s

(b) The instantaneous velocity when $t = 4$ (h approaches 0) is -128 ft/s.

6. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{avg}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{avg}} = 4.42 \text{ m/s}$

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{avg}} = 5.35 \text{ m/s}$

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{avg}} = 6.094 \text{ m/s}$

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{avg}} = 6.2614 \text{ m/s}$

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{avg}} = 6.27814 \text{ m/s}$

(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m/s.

7. (a) (i) On the interval $[2, 4]$, $v_{\text{avg}} = \dfrac{s(4) - s(2)}{4 - 2} = \dfrac{79.2 - 20.6}{2} = 29.3 \text{ ft/s}$.

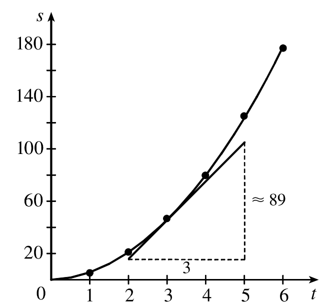
(ii) On the interval $[3, 4]$, $v_{\text{avg}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{79.2 - 46.5}{1} = 32.7 \text{ ft/s}$.

(iii) On the interval $[4, 5]$, $v_{\text{avg}} = \dfrac{s(5) - s(4)}{5 - 4} = \dfrac{124.8 - 79.2}{1} = 45.6 \text{ ft/s}$.

(iv) On the interval $[4, 6]$, $v_{\text{avg}} = \dfrac{s(6) - s(4)}{6 - 4} = \dfrac{176.7 - 79.2}{2} = 48.75 \text{ ft/s}$.

(b) Using the points $(2, 16)$ and $(5, 105)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about

$$\frac{105 - 16}{5 - 2} = \frac{89}{3} \approx 29.7 \text{ ft/s}.$$



8. (a) (i) $s = s(t) = 2\sin \pi t + 3\cos \pi t$. On the interval $[1, 2]$, $v_{\text{avg}} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{3 - (-3)}{1} = 6 \text{ cm/s}$.

(ii) On the interval $[1, 1.1]$, $v_{\text{avg}} = \dfrac{s(1.1) - s(1)}{1.1 - 1} \approx \dfrac{-3.471 - (-3)}{0.1} = -4.71 \text{ cm/s}$.

(iii) On the interval $[1, 1.01]$, $v_{\text{avg}} = \dfrac{s(1.01) - s(1)}{1.01 - 1} \approx \dfrac{-3.0613 - (-3)}{0.01} = -6.13 \text{ cm/s}$.

(iv) On the interval $[1, 1.001]$, $v_{\text{avg}} = \dfrac{s(1.001) - s(1)}{1.001 - 1} \approx \dfrac{-3.00627 - (-3)}{0.001} = -6.27 \text{ cm/s}$.

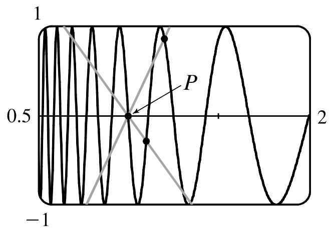
(b) The instantaneous velocity of the particle when $t = 1$ appears to be about -6.3 cm/s.

9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b) 

We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

1.5 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) As x approaches 2 from the left, the values of $f(x)$ approach 3, so $\lim_{x \to 2^-} f(x) = 3$.

(b) As x approaches 2 from the right, the values of $f(x)$ approach 1, so $\lim_{x \to 2^+} f(x) = 1$.

(c) $\lim_{x \to 2} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(d) When $x = 2$, $y = 3$, so $f(2) = 3$.

(e) As x approaches 4, the values of $f(x)$ approach 4, so $\lim_{x \to 4} f(x) = 4$.

(f) There is no value of $f(x)$ when $x = 4$, so $f(4)$ does not exist.

5. (a) As x approaches 1, the values of $f(x)$ approach 2, so $\lim_{x \to 1} f(x) = 2$.

(b) As x approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim_{x \to 3^-} f(x) = 1$.

(c) As x approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim_{x \to 3^+} f(x) = 4$.

(d) $\lim_{x \to 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(e) When $x = 3$, $y = 3$, so $f(3) = 3$.

6. (a) $h(x)$ approaches 4 as x approaches -3 from the left, so $\lim_{x\to -3^-} h(x) = 4$.

(b) $h(x)$ approaches 4 as x approaches -3 from the right, so $\lim_{x\to -3^+} h(x) = 4$.

(c) $\lim_{x\to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) $h(-3)$ is not defined, so it doesn't exist.

(e) $h(x)$ approaches 1 as x approaches 0 from the left, so $\lim_{x\to 0^-} h(x) = 1$.

(f) $h(x)$ approaches -1 as x approaches 0 from the right, so $\lim_{x\to 0^+} h(x) = -1$.

(g) $\lim_{x\to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

(h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of h.

(i) Since $\lim_{x\to 2^-} h(x) = 2$ and $\lim_{x\to 2^+} h(x) = 2$, we have $\lim_{x\to 2} h(x) = 2$.

(j) $h(2)$ is not defined, so it doesn't exist.

(k) $h(x)$ approaches 3 as x approaches 5 from the right, so $\lim_{x\to 5^+} h(x) = 3$.

(l) $h(x)$ does not approach any one number as x approaches 5 from the left, so $\lim_{x\to 5^-} h(x)$ does not exist.

7. (a) $\lim_{x\to 4^-} g(x) \neq \lim_{x\to 4^+} g(x)$, so $\lim_{x\to 4} g(x)$ does not exist. However, there is a point on the graph representing $g(4)$. Thus, $a = 4$ satisfies the given description.

(b) $\lim_{x\to 5^-} g(x) = \lim_{x\to 5^+} g(x)$, so $\lim_{x\to 5} g(x)$ exists. However, $g(5)$ is not defined. Thus, $a = 5$ satisfies the given description.

(c) From part (a), $a = 4$ satisfies the given description. Also, $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$ exist, but $\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$. Thus, $\lim_{x\to 2} g(x)$ does not exist, and $a = 2$ also satisfies the given description.

(d) $\lim_{x\to 4^+} g(x) = g(4)$, but $\lim_{x\to 4^-} g(x) \neq g(4)$. Thus, $a = 4$ satisfies the given description.

8. (a) $\lim_{x\to -3} A(x) = \infty$

(b) $\lim_{x\to 2^-} A(x) = -\infty$

(c) $\lim_{x\to 2^+} A(x) = \infty$

(d) $\lim_{x\to -1} A(x) = -\infty$

(e) The equations of the vertical asymptotes are $x = -3$, $x = -1$ and $x = 2$.

9. (a) $\lim_{x\to -7} f(x) = -\infty$

(b) $\lim_{x\to -3} f(x) = \infty$

(c) $\lim_{x\to 0} f(x) = \infty$

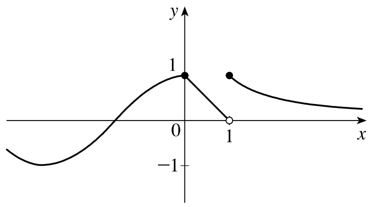
(d) $\lim_{x\to 6^-} f(x) = -\infty$

(e) $\lim_{x\to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

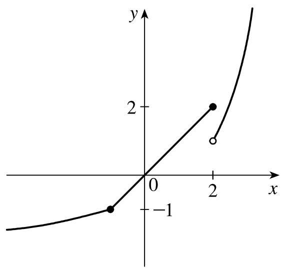
10. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11.



From the grapn of f we see that $\lim_{x\to 1^-} f(x) = 0$, but $\lim_{x\to 1^+} f(x) = 1$, so $\lim_{x\to a} f(x)$ does not exist for $a = 1$. However, $\lim_{x\to a} f(x)$ exists for all other values of a. Thus, $\lim_{x\to a} f(x)$ exists for all a in $(-\infty, 1) \cup (1, \infty)$.

12.

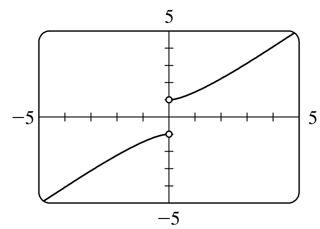


From the graph of f we see that $\lim_{x\to 2^-} f(x) = 2$, but $\lim_{x\to 2^+} f(x) = 1$, so $\lim_{x\to a} f(x)$ does not exist for $a = 2$. However, $\lim_{x\to a} f(x)$ exists for all other values of a. Thus, $\lim_{x\to a} f(x)$ exists for all a in $(-\infty, 2) \cup (2, \infty)$.

13. (a) From the graph, $\lim_{x\to 0^-} f(x) = -1$.

(b) From the graph, $\lim_{x\to 0^+} f(x) = 1$.

(c) Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.

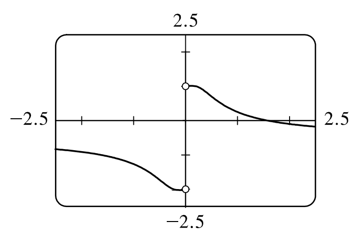


$f(x) = x\sqrt{1 + x^{-2}}$

14. (a) From the graph, $\lim_{x\to 0^-} f(x) = -2$.

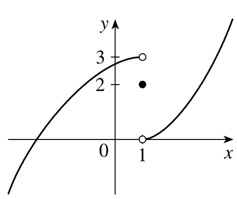
(b) From the graph, $\lim_{x\to 0^+} f(x) = 1$.

(c) Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.



$f(x) = \dfrac{3^{1/x} - 2}{3^{1/x} + 1}$

15. $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 0$, $f(1) = 2$



16. $\lim_{x\to 0} f(x) = 4$, $\lim_{x\to 8^-} f(x) = 1$, $\lim_{x\to 8^+} f(x) = -3$, $f(0) = 6$, $f(8) = -1$