CONTENTS

1 VECTORS AND KINEMATICS

1.1 *Vector algebra 1*

$\mathbf{A} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) \quad \mathbf{B} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(a) $\mathbf{A} + \mathbf{B} = (2+5)\hat{\mathbf{i}} + (-3+1)\hat{\mathbf{j}} + (7+2)\hat{\mathbf{k}} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

(b) $\mathbf{A} - \mathbf{B} = (2-5)\hat{\mathbf{i}} + (-3-1)\hat{\mathbf{j}}(7-2)\hat{\mathbf{k}} = -3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

(c) $\mathbf{A} \cdot \mathbf{B} = (2)(5) + (-3)(1) + (7)(2) = 21$

(d) $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 7 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= -13\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 17\hat{\mathbf{k}}$$

1.2 *Vector algebra 2*

$\mathbf{A} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \quad \mathbf{B} = (6\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

(a) $A^2 = \mathbf{A} \cdot \mathbf{A} = 3^2 + (-2)^2 + 5^2 = 38$

(b) $B^2 = \mathbf{B} \cdot \mathbf{B} = 6^2 + (-7)^2 + 4^2 = 101$

(c) $(\mathbf{A} \cdot \mathbf{B})^2 = [(3)(6) + (-2)(-7) + (5)(4)]^2 = [18 + 14 + 20]^2 = 52^2 = 2704$

1.3 *Cosine and sine by vector algebra*

$\mathbf{A} = (3\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad \mathbf{B} = (-2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

(a)

$$\mathbf{A}\cdot\mathbf{B} = A\,B\,\cos(\mathbf{A},\mathbf{B})$$

$$\cos(\mathbf{A},\mathbf{B}) = \frac{\mathbf{A}\cdot\mathbf{B}}{A\,B} = \frac{(-6+1+1)}{\sqrt{(9+1+1)}\,\sqrt{4+1+1)}} = \frac{-4}{\sqrt{11}\,\sqrt{6}} \approx 0.492$$

(b) *method 1:*

$$|\mathbf{A}\times\mathbf{B}| = A\,B\sin(\mathbf{A},\mathbf{B})$$

$$\sin(\mathbf{A},\mathbf{B}) = \frac{|\mathbf{A}\times\mathbf{B}|}{A\,B}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 1 \\ -2 & 1 & 1 \end{vmatrix} = (1-1)\,\hat{\mathbf{i}} - (3+2)\,\hat{\mathbf{j}} + (3+2)\,\hat{\mathbf{k}} = -5\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$$

$$|\mathbf{A}\times\mathbf{B}| = \sqrt{5^2+5^2} = 5\,\sqrt{2}$$

$$\sin(\mathbf{A},\mathbf{B}) = \frac{|\mathbf{A}\times\mathbf{B}|}{A\,B} = \frac{5\,\sqrt{2}}{\sqrt{11}\,\sqrt{6}} \approx 0.870$$

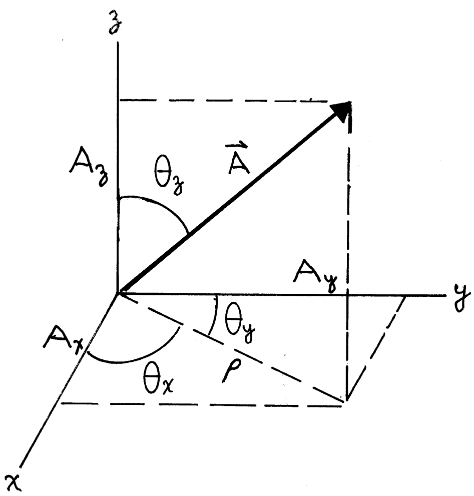
(c) *method 2 (simpler) – use:*

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(\mathbf{A},\mathbf{B}) = \sqrt{1-\cos^2(\mathbf{A},\mathbf{B})} = \sqrt{1-(0.492)^2} \quad \text{from (a)} \approx 0.871$$

1.4 *Direction cosines*

Note that here α, β, γ stand for direction cosines, not for the angles shown in the figure:
$\theta_x = \cos^{-1}\alpha$,
$\theta_y = \cos^{-1}\beta$,
$\theta_z = \cos^{-1}\gamma$.



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$$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$$
$$A_x = \mathbf{A}\cdot\hat{\mathbf{i}} = A\,\cos(\mathbf{A},\hat{\mathbf{i}}) \equiv A\,\alpha$$
$$\alpha = \cos(\mathbf{A},\hat{\mathbf{i}}) = \cos\theta_x.$$

Similarly,

$$A_y = A\,\cos(\mathbf{A},\hat{\mathbf{j}}) \equiv A\,\beta$$
$$\beta = \cos(\mathbf{A},\hat{\mathbf{j}}) = \cos\theta_y$$
$$A_z = A\,\cos(\mathbf{A},\hat{\mathbf{k}}) \equiv A\,\gamma$$
$$\gamma = \cos(\mathbf{A},\hat{\mathbf{k}}) = \cos\theta_z$$

Using these results,

$$A^2 = A_x^2 + A_y^2 + A_z^2$$
$$= A^2\,(\alpha^2 + \beta^2 + \gamma^2)$$

from which it follows that

$\alpha^2 + \beta^2 + \gamma^2 = 1$

Another way to see this is

$$A^2 = \rho^2 + A_z^2 = A_x^2 + A_y^2 + A_z^2 = A^2\,(\alpha^2 + \beta^2 + \gamma^2)$$

and it follows as before that

$\alpha^2 + \beta^2 + \gamma^2 = 1.$

1.5 *Perpendicular vectors*

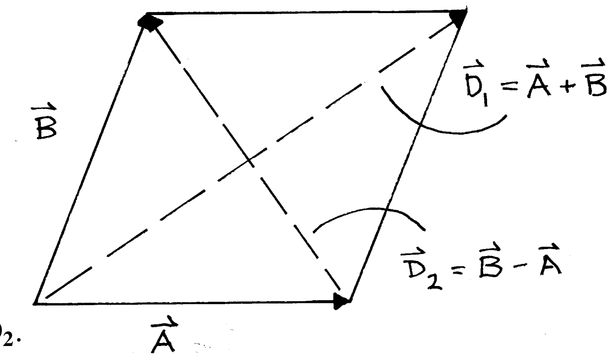
Given $|\mathbf{A}-\mathbf{B}| = |\mathbf{A}+\mathbf{B}|$ with $\mathbf{A}$ and $\mathbf{B}$ nonzero. Evaluate the magnitudes by squaring.

$$A^2 - 2\,\mathbf{A}\cdot\mathbf{B} + B^2 = A^2 + 2\,\mathbf{A}\cdot\mathbf{B} + B^2$$
$$-2\,\mathbf{A}\cdot\mathbf{B} = +2\,\mathbf{A}\cdot\mathbf{B}.$$
$$\mathbf{A}\cdot\mathbf{B} = 0$$

and it follows that $\mathbf{A} \perp \mathbf{B}$.

1.6 *Diagonals of a parallelogram*

The parallelogram is equilateral, so $A = B$.

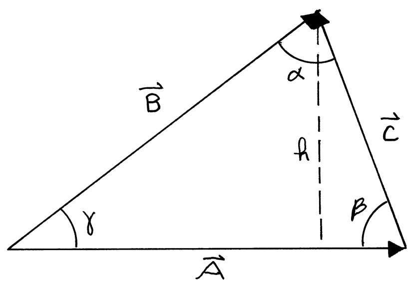


$$\mathbf{D_1} = \mathbf{A} + \mathbf{B}$$
$$\mathbf{D_2} = \mathbf{B} - \mathbf{A}$$
$$\mathbf{D_1} \cdot \mathbf{D_2} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{B} - \mathbf{A}) = A^2 - B^2 = 0.$$

Hence $\mathbf{D_1} \cdot \mathbf{D_2} = \mathbf{0}$ and it follows that $\mathbf{D_1} \perp \mathbf{D_2}$.

1.7 *Law of sines*

The area $\mathcal{A}$ of the triangle is



$$\mathcal{A} = \frac{1}{2}A\,h = \frac{1}{2}A\,B\,\sin\gamma = \frac{1}{2}|\mathbf{A} \times \mathbf{B}|$$

Similarly,

$$\mathcal{A} = \frac{1}{2}|\mathbf{B} \times \mathbf{C}| = \frac{1}{2}BC\sin\alpha$$
$$\mathcal{A} = \frac{1}{2}|\mathbf{C} \times \mathbf{A}| = \frac{1}{2}AC\sin\beta.$$

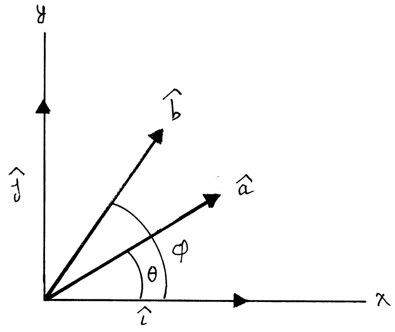
Hence $AB\,\sin\gamma = BC\,\sin\alpha = AC\,\sin\beta$, from which it follows

$$\frac{\sin\gamma}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$

Introducing the cross product makes the notation convenient, and emphasizes the relation between the cross product and the area of the triangle, but it is not essential for the proof.

1.8 *Vector proof of a trigonometric identity*

Given two unit vectors $\hat{\mathbf{a}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ and $\hat{\mathbf{b}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$, with $a = 1, b = 1$. First evaluate their scalar product using components:



$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= ab\ \cos\theta\cos\phi + ab\ \sin\theta\sin\phi\\ &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

then evaluate their scalar product geometrically.

$$\mathbf{a}\cdot\mathbf{b} = ab\ \cos(\mathbf{a},\mathbf{b})\ = ab\cos(\phi-\theta) = \cos(\phi-\theta)$$

Equating the two results,

$$\cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$

1.9 *Perpendicular unit vector*

Given $\mathbf{A} = (\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$ and $\mathbf{B} = (2\,\hat{\mathbf{i}}+\hat{\mathbf{j}}-3\,\hat{\mathbf{k}})$, find $\mathbf{C}$ such that $\mathbf{A}\cdot\mathbf{C} = \mathbf{0}$ and $\mathbf{B}\cdot\mathbf{C} = \mathbf{0}$.

$$\begin{aligned}\mathbf{C} &= C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}}\\ &= C_x(\hat{\mathbf{i}} + (C_y/C_x)\,\hat{\mathbf{j}} + (C_z/C_x)\,\hat{\mathbf{k}})\\ \mathbf{A}\cdot\mathbf{C} &= C_x(1 + (C_y/C_x) - (C_z/C_x)) = 0\\ \mathbf{B}\cdot\mathbf{C} &= C_x(2 + (C_y/C_x) - 3(C_z/C_x)) = 0\end{aligned}$$

We have two equations for the two unknowns (C_y/C_x) and (C_z/C_x).

$$\begin{aligned}1 + (C_y/C_x) - (C_z/C_x) &= 0\\ 2 + (C_y/C_x) - 3(C_z/C_x) &= 0.\end{aligned}$$

The solutions are $(C_y/C_x) = -\frac{1}{2}$ and $(C_z/C_x) = \frac{1}{2}$, so that $\mathbf{C} = \mathbf{C_x}(\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}})$. To evaluate C_x, apply the condition that $\mathbf{C}$ is a unit vector.

$$\begin{aligned}C^2 &= \frac{3}{2}C_x^2\ = 1\\ C_x &= \pm\sqrt{(2/3)}\\ \hat{\mathbf{C}} &= \pm\sqrt{(2/3)}\,(\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}})\end{aligned}$$

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which can be written

$$\hat{\mathbf{C}} = \pm\frac{1}{\sqrt{6}}(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Geometrically, **C** can be perpendicular to both **A** and **B** only if **C** is perpendicular to the plane determined by **A** and **B**. From the standpoint of vector algebra, this implies that $\mathbf{C} \propto \mathbf{A}\times\mathbf{B}$. To prove this, evaluate $\mathbf{A}\times\mathbf{B}$.

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 1 & -1\\ 2 & 1 & -3\end{vmatrix}\\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}\\ &\propto \mathbf{C}.\end{aligned}$$

1.10 *Perpendicular unit vectors*

Given $\mathbf{A} = \mathbf{3\hat{i}} + \mathbf{4\hat{j}} - \mathbf{4\hat{k}}$, find a unit vector $\hat{\mathbf{B}}$ perpendicular to **A**.

(a)

$$\begin{aligned}\mathbf{B} &= B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} \;= B_x[\hat{\mathbf{i}} + (B_y/B_x)\hat{\mathbf{j}}]\\ \mathbf{A}\cdot\mathbf{B} &= B_x[3 + 4(B_y/B_x)] \;= 0\\ B_y/B_x &= -3/4\\ \mathbf{B} &= B_x[\hat{\mathbf{i}} - \frac{3}{4}\hat{\mathbf{j}}]\end{aligned}$$

To evaluate B_x, note that **B** is a unit vector, $B^2 = 1$.

$$1 = B_x^2\left[(1)^2 + \left(\frac{3}{4}\right)^2\right] = \left(\frac{25}{16}\right)B_x^2$$

which gives

$$\begin{aligned}B_x &= \pm(4/5)\\ \hat{\mathbf{B}} &= \pm(4/5)(\hat{\mathbf{i}} - (3/4)\hat{\mathbf{j}}) \;= \pm\frac{1}{5}(4\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}})\end{aligned}$$

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(b)

$$\begin{aligned}\mathbf{C} &= C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}}\\ &= C_x[\hat{\mathbf{i}} + (C_y/C_x)\,\hat{\mathbf{j}} + (C_z/C_x)\,\hat{\mathbf{k}}]\end{aligned}$$

$$\mathbf{A}\cdot\mathbf{C} = 0 \;\Rightarrow\; C_x[3 + 4(C_y/C_x) - 4(C_z/C_x)] = 0$$

$$\mathbf{B}\cdot\mathbf{C} = 0 \;\Rightarrow\; \frac{1}{5}C_x[4 - 3(C_y/C_x)] = 0$$

$$C_y/C_x = 4/3 \quad C_z/C_x = 25/12$$

To make **C** a unit vector,

$$C^2 = C_x^2\left[(1)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{25}{12}\right)^2\right] = 1$$

$$C_x \approx \pm 0.348$$

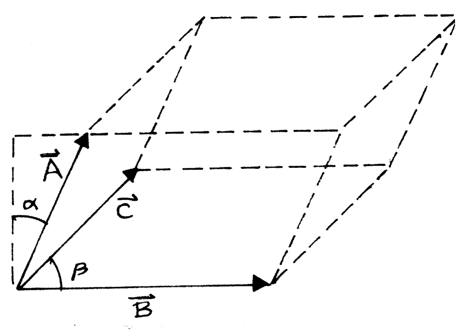
(c) The vector **B** × **C** is perpendicular (normal) to the plane defined by **B** and **C**, so we want to prove

$$\mathbf{A} \propto \mathbf{B}\times\mathbf{C}$$

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= C_x\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{4}{5} & -\frac{3}{5} & 0\\ 1 & \frac{4}{3} & \frac{25}{12}\end{vmatrix}\\ &= C_x\left[-\left(\frac{75}{60}\right)\hat{\mathbf{i}} - \left(\frac{100}{60}\right)\hat{\mathbf{j}} + \left(\frac{25}{15}\right)\hat{\mathbf{k}}\right]\\ &= \left(\frac{5}{12}\right)C_x(-3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}) \;\propto \mathbf{A}.\end{aligned}$$

1.11 *Volume of a parallelepiped*

With reference to the sketch, the height is $A\cos\alpha$, so the frontal area is $AB\cos\alpha$. The depth is $C\sin\beta$, so the volume V is



$$V = (AB\cos\alpha)(C\sin\beta) \;= (A\cos\alpha)(BC\sin\beta) \;= \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

The same approach can be used starting with a different face.

$$V = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \quad V = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

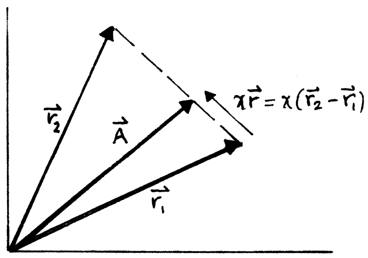
Note that **A**, **B**, **C** are arbitrary vectors. This proves the vector identity

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

1.12 *Constructing a vector to a point*

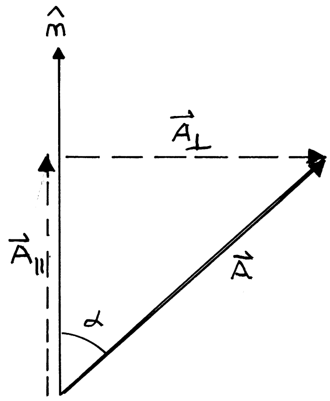
Applying vector addition to the lower triangle in the sketch,

$$\mathbf{A} = \mathbf{r}_1 + x(\mathbf{r}_2 - \mathbf{r}_1)$$
$$= (1 - x)\mathbf{r}_1 + x\mathbf{r}_2$$



1.13 *Expressing one vector in terms of another*

We will express vector **A** in terms of a unit vector $\hat{\mathbf{n}}$. As shown in the sketch, we can write **A** as the vector sum of a vector $\mathbf{A}_{\|}$ parallel to $\hat{\mathbf{n}}$ and a vector $\mathbf{A}_{\perp}$ perpendicular to $\hat{\mathbf{n}}$, so that $\mathbf{A} = \mathbf{A}_{\|} + \mathbf{A}_{\perp}$.



$$|\mathbf{A}_{\|}| = A\cos\alpha$$

The direction of $\mathbf{A}_{\|}$ is along $\hat{\mathbf{n}}$, so it follows that

$$\mathbf{A}_{\|} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}.$$
$$|\mathbf{A}_{\perp}| = A\sin\alpha \;\; = |\hat{\mathbf{n}}\times\mathbf{A}|$$

The direction of $(\hat{\mathbf{n}}\times\mathbf{A})$ is into the paper, so taking its cross product with $\hat{\mathbf{n}}$ gives a vector $(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$ along $\mathbf{A}_{\perp}$ and with the correct magnitude. Hence

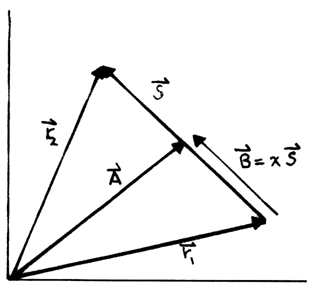
$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$$

1.14 *Two points*

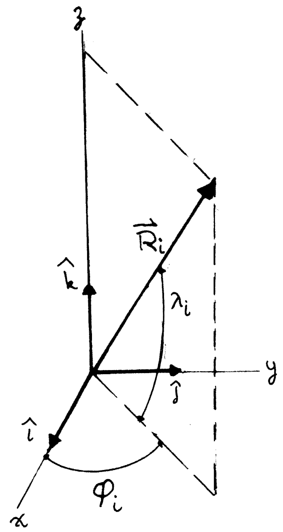
$$\mathbf{S} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathbf{B} = x\mathbf{S} \qquad \mathbf{A} = \mathbf{r}_1 + \mathbf{B}$$

$x = 0$ at $t = 0$; $x = 1$ at $t = T$
so that $x = t/T$, linear in t

$$\mathbf{A} = \mathbf{r}_1 + x\mathbf{S} \;\; = \mathbf{r}_1 + \frac{t}{T}(\mathbf{r}_2 - \mathbf{r}_1) \;\; = \left(1 - \frac{t}{T}\right)\mathbf{r}_1 + \frac{t}{T}\mathbf{r}_2$$



1.15 *Great circle*



Consider vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ from the center of a sphere of radius R to points on the surface. To avoid complications, the sketch shows the geometry of a generic vector $\mathbf{R_i}$ (i = 1 or 2) making angles λ_i and ϕ_i. The magnitude of $\mathbf{R_i}$ is R, so $R_1 = R_2 = R$. The coordinates of a point on the surface are

$$\mathbf{R}_i = R\cos\lambda_i\cos\phi_i\,\hat{\mathbf{i}} + R\cos\lambda_i\sin\phi_i\,\hat{\mathbf{j}} + R\sin\lambda_i\,\hat{\mathbf{k}}$$

The angle between two points can be found using the dot product.

$$\theta(1,2) = \arccos\left(\frac{\mathbf{R_1}\cdot\mathbf{R_2}}{R_1R_2}\right) = \arccos\left(\frac{\mathbf{R_1}\cdot\mathbf{R_2}}{R^2}\right)$$

Note that $\theta(1,2)$ is in radians.

The great circle distance between $\mathbf{R_1}$ and $\mathbf{R_2}$ is $S = R\theta(1,2)$.

$$\mathbf{R_1}\cdot\mathbf{R_2} = R^2(\cos\lambda_1\cos\phi_1\cos\lambda_2\cos\phi_2 + \cos\lambda_1\sin\phi_1\cos\lambda_2\sin\phi_2 + \sin\lambda_1\sin\lambda_2)$$

Hence

$$\begin{aligned}
S &= R\,\theta(1,2)\\
&= R\arccos\left[\cos\lambda_1\cos\lambda_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + \sin\lambda_1\sin\lambda_2\right]\\
&= R\arccos\left\{\frac{1}{2}\cos(\lambda_1+\lambda_2)\left[\cos(\phi_1-\phi_2)-1\right] + \frac{1}{2}\cos(\lambda_1-\lambda_2)\left[\cos(\phi_1-\phi_2)+1\right]\right\}
\end{aligned}$$