Contents

Chapter 1

Review of Sequences and Infinite Series

1. For those sequences that converge, find the limit $\lim_{n\to\infty} a_n$.

a. $a_n = \dfrac{n^2+1}{n^3+1}$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2+1}{n^3+1} \\ &= \lim_{n\to\infty} \frac{n^2}{n^3} \\ &= \lim_{n\to\infty} \frac{1}{n} = 0.\end{aligned}$$

b. $a_n = \dfrac{3n+1}{n+2}$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3n+1}{n+2} \\ &= \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{2}{n}} = 3.\end{aligned}$$

c. $a_n = \left(\dfrac{3}{n}\right)^{1/n}$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} \\ &= \lim_{n\to\infty} \left(\frac{3^{1/n}}{n^{1/n}}\right) = 1.\end{aligned}$$

d. $a_n = \dfrac{2n^2+4n^3}{n^3+5\sqrt{2+n^6}}$.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2n^2+4n^3}{n^3+5\sqrt{2+n^6}} \\ &= \lim_{n\to\infty} \frac{\frac{2}{n}+4}{1+5\sqrt{\frac{2}{n^6}+1}} \\ &= \lim_{n\to\infty} \frac{4}{1+5} = \frac{2}{3}.\end{aligned}$$

e. $a_n = n \ln\left(1 + \frac{1}{n}\right).$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)\\ &= \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n\\ &= \ln e = 1.\end{aligned}$$

f. $a_n = n \sin\left(\frac{1}{n}\right).$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\\ &= \lim_{x\to 0} \frac{\sin x}{x} = 1.\end{aligned}$$

g. $a_n = \frac{(2n+3)!}{(n+1)!}.$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(2n+3)!}{(n+1)!}\\ &= \lim_{n\to\infty} \frac{(2n+3)(2n+2)\cdots(n+2)(n+1)!}{(n+1)!}\\ &= \infty\end{aligned}$$

2. Find the sum for each of the series:

a. $\sum_{n=0}^{\infty} (-1)^n \frac{3}{4^n}.$

This is a geometric series with $a = 3$ and $r = -\frac{1}{4}$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=0}^{\infty} 3\left(-\frac{1}{4}\right)^n = \frac{3}{1+\frac{1}{4}} = \frac{12}{5}.$$

b. $\sum_{n=2}^{\infty} \frac{2}{5^n}.$

This is a geometric series with $a = \frac{2}{5^2}$ and $r = \frac{1}{5}$.

$$\sum_{n=2}^{\infty} \frac{2}{5^n} = \frac{2/25}{1-\frac{1}{5}} = \frac{1}{10}.$$

c. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right).$

This is the sum of two geometric series with $a = 5$, $r = \frac{1}{2}$ and $a = 1$, $r = \frac{1}{3}$.

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right) = \frac{5}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = \frac{23}{2}.$$

d. $\sum_{n=2}^{\infty} e^{-2ns}$, for $s > 0$.

This is a geometric series and can be summed.

$$\sum_{n=2}^{\infty} e^{-2ns} = \sum_{n=2}^{\infty} (e^{-2s})^n = \frac{e^{-4s}}{1+e^{-2s}} = \frac{1}{2} e^{-3s} \operatorname{sech} s.$$

e. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

Using a partial fraction decomposition, this can be written as a telescoping series. The N-th partial sum is

$$\begin{aligned}
\sum_{n=1}^{N} \frac{3}{n(n+3)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+3} \right) \\
&= \left(1 - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) \\
&\quad + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) + \cdots + \left(\cancel{\frac{1}{N-2}} - \frac{1}{N+1}\right) \\
&\quad + \left(\cancel{\frac{1}{N-1}} - \frac{1}{N+2}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+3}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} \right).
\end{aligned}$$

Here forward crossed terms like $\cancel{\frac{1}{4}}$ cancel with terms later in sum and backward crossed terms like $\bcancel{\frac{1}{4}}$ cancel with earlier terms.

Letting $N \to \infty$, we have $\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \frac{11}{6}$.

f. $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

One can factor the nth term and use partial fraction decomposition. This yields

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2} \sum_{n=1}^{N} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \\
&= \frac{1}{2} \left[\left(1 - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right) \right] \\
&= \frac{1}{2} \left[1 - \frac{1}{2N+1} \right].
\end{aligned}$$

Letting $N \to \infty$, we have $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

3. Sum the geometric progression,

$$\sum_{n=-N}^{N} e^{in\omega}.$$

The sum of a geometric progression takes the form

$$\sum_{n=0}^{N-1} ar^n = \frac{a(1-r^n)}{1-r}.$$

Re-indexing the sum, using $j = n + N$, or writing out the terms, we have

$$\begin{aligned}
\sum_{n=-N}^{N} e^{in\omega} &= e^{-iN\omega} + e^{-i(N-1)\omega} + \cdots + e^{i(N-1)\omega} + e^{iN\omega} \\
&= e^{-iN\omega}\left[1 + e^{i\omega} + \cdots + e^{i(2N-1)\omega} + e^{i2N\omega}\right] \\
&= e^{-iN\omega}\sum_{j=0}^{2N}\left(e^{i\omega}\right)^j \\
&= e^{-iN\omega}\frac{1 - e^{i(2N+1)\omega}}{1 - e^{i\omega}} \\
&= \frac{e^{-iN\omega} - e^{i(N+1)\omega}}{1 - e^{i\omega}} \\
&= \frac{e^{-iN\omega} - e^{i(N+1)\omega}}{(e^{-i\omega/2} - e^{i\omega/2})e^{i\omega/2}} \\
&= \frac{e^{-i(N+1/2)\omega} - e^{i(N+1/2)\omega}}{e^{-i\omega/2} - e^{i\omega/2}} \\
&= \frac{\sin(N+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}.
\end{aligned}$$

4. Determine if the following converge, or diverge, using one of the convergence tests. If the series converges, is it absolute or conditional?

a. $\displaystyle\sum_{n=1}^{\infty}\frac{n+4}{2n^3+1}$.

Since the tail of the series determines the convergence, we note that for large n

$$\frac{n+4}{2n^3+1} \sim \frac{n}{2n^3} = \frac{1}{2n^2}.$$

So, we can compare the original series to $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

We compute the following limit

$$\begin{aligned}
\lim_{n\to\infty}\frac{\frac{n+4}{2n^3+1}}{\frac{1}{n^2}} &= \lim_{n\to\infty}\frac{2n^3+1}{n^2(n+4)} \\
&= \lim_{n\to\infty}\frac{(2n^3}{n^3} = 2.
\end{aligned}$$

Therefore, these series either both converge or both diverge. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, then the original series converges by the Limit Comparison Test. Also, since the terms are all positive, it converges absolutely.

b. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$.

We note that

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| \le \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Therefore, this series converges absolutely.

c. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

We apply the nth Root Test.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\ &= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.\end{aligned}$$

Therefore, the series converges by the nth Root Test. Also, since the terms are all positive, it converges absolutely.

d. $\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{2n^2-3}$.

We first note that this is an alternating series whose terms have magnitude decreasing to zero,

$$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{2n^2-3} = 0 + \frac{1}{5} - \frac{2}{15} + \frac{3}{38} - \dots.$$

Therefore, the series converges by the Liebniz Test.

Dropping the signs, we have that $\sum_{n=1}^{\infty} \frac{n-1}{2n^2-3}$ has terms that behave like

$$\frac{n-1}{2n^2-3} \sim \frac{1}{2n}.$$

Therefore, this series behaves like the harmonic series, which diverges. So, the original series converges conditionally.

e. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

We note that $\frac{\ln n}{n} > \frac{1}{n}$, $n \geq 3$. Therefore,

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} > \sum_{n=1}^{\infty} \frac{1}{n} > \infty.$$

Therefore, this series diverges by the Comparison Test.

f. $\sum_{n=1}^{\infty} \frac{100^n}{n^{200}}$.

We apply the nth Root Test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{100}{1} = 100 > 1$$

Therefore, the series diverges by the nth Root Test.

g. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+3}$.

For this series the terms do not go to zero for large n. Namely,

$$\lim_{n\to\infty} \frac{n}{n+3} = 1.$$

So, by the nth term divergence test, this series diverges.

h. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{5n}}{n+1}$.

The magnitudes of the terms goes to zero for large n.

$$\lim_{n\to\infty} \frac{\sqrt{5n}}{n+1} = \lim_{n\to\infty} \frac{\sqrt{5}}{\sqrt{n}} = 0.$$

Therefore, the series converges by the Liebniz Test.

Dropping the signs, we have that $\sum_{n=1}^{\infty} \frac{\sqrt{5n}}{n+1}$ has terms that behave like

$$\frac{\sqrt{5n}}{n+1} \sim \frac{\sqrt{5}}{\sqrt{n}}.$$

Therefore, this series diverges according to the p test. So, the original series converges conditionally.

5. Do the following:

a. Compute: $\lim_{n\to\infty} n \ln\left(1 - \frac{3}{n}\right)$.

$$\begin{aligned}\lim_{n\to\infty} n \ln\left(1 - \frac{3}{n}\right) &= \lim_{n\to\infty} \ln\left(1 - \frac{3}{n}\right)^n \\ &= \ln e^{-3} = -3.\end{aligned}$$

b. Use L'Hopital's Rule to evaluate $L = \lim_{x\to\infty} \left(1 - \frac{4}{x}\right)^x$. [Hint: Consider $\ln L$.]

We use $\ln L$ and L'Hopital's Rule to find

$$\begin{aligned}\ln L &= \lim_{x\to\infty} \ln\left(1 - \frac{4}{x}\right)^x \\ &= \lim_{x\to\infty} x \ln\left(1 - \frac{4}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln(1-\frac{4}{x})}{\frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\frac{4}{x^2}}{-\frac{1}{x^2}(1-\frac{4}{x})} \\ &= \lim_{x\to\infty} \frac{4}{-(1-\frac{4}{x})} = -4.\end{aligned}$$

Since $\ln L = -4$, $L = e^{-4}$.

c. Determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{3n+2}\right)^{n^2}$.

We apply the nth Root Test.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \left(\frac{n}{3n+2}\right)^n \\ &= \lim_{n\to\infty} \frac{1}{(3+\frac{2}{n})^n} \\ &= \lim_{n\to\infty} \frac{1}{3^n\left(1+\frac{2/3}{n}\right)^n} = 0 < 1.\end{aligned}$$

Therefore, by the nth Root Test this series converges.

d. Sum the series $\sum_{n=1}^{\infty}\left[\tan^{-1} n - \tan^{-1}(n+1)\right]$ by first writing the Nth partial sum and then computing $\lim_{N\to\infty} s_N$.

The Nth partial sum is

$$\begin{aligned} s_N &= \sum_{n=1}^{N}\left[\tan^{-1} n - \tan^{-1}(n+1)\right] \\ &= \left(\tan^{-1} 1 - \cancel{\tan^{-1} 2}\right) + \left(\cancel{\tan^{-1} 2} - \cancel{\tan^{-1} 3}\right) \\ &\quad + \cdots + \left(\cancel{\tan^{-1} N} - \tan^{-1}(N+1)\right) \\ &= \tan^{-1} 2 - \tan^{-1}(N+1) \end{aligned}$$

Letting $N \to \infty$, we have

$$\sum_{n=1}^{\infty}\left[\tan^{-1} n - \tan^{-1}(n+1)\right] = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}.$$

6. Consider the sum $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)}$.

a. Use an appropriate convergence test to show that this series converges.

Since the tail of the series determines the convergence, we note that for large n

$$\frac{1}{(n+2)(n+1)} \sim \frac{1}{n^2}.$$

So, we can compare the original series to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the latter series converges, so does this one by the Limit Comparison Test.

One can also use the following

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, the series converges by the Comparison Test as well.

b. Verify that

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2} - \frac{n}{n+1}\right).$$

One just adds the terms to verify the sum.

$$\begin{aligned} \frac{n+1}{n+2} - \frac{n}{n+1} &= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} \\ &= \frac{1}{(n+2)(n+1)}. \end{aligned}$$

Note that partial fractions does not give this representation, but instead gives

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right).$$

This may also be summed as a telescoping series but the instructions for part c apply to the other representation.

c. Find the nth partial sum of the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right)$ and use it to determine the sum of the resulting *telescoping* series.

$$\begin{aligned} s_N &= \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right) \\ &= \left(\frac{2}{3}-\frac{1}{2}\right)+\left(\frac{3}{4}-\frac{2}{3}\right)+\left(\frac{4}{5}-\frac{3}{4}\right)+\cdots+\left(\frac{N+1}{N+2}-\frac{N}{N+1}\right) \\ &= -\frac{1}{2}+\frac{N+1}{N+2}. \end{aligned}$$

Here forward crossed terms like $\frac{2}{3}$ cancel with terms later in sum and backward crossed terms like $\frac{2}{3}$ cancel with earlier terms.

Letting $N\to\infty$, we have $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right)=-\frac{1}{2}+1=\frac{1}{2}$.

7. Recall that the alternating harmonic series converges conditionally.

a. From the Taylor series expansion for $f(x)=\ln(1+x)$, inserting $x=1$ gives the alternating harmonic series. What is the sum of the alternating harmonic series?

The Taylor series expansion for $f(x)=\ln(1+x)$ is given by

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}.$$

Inserting $x=1$ gives the sum of the alternating harmonic series

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}.$$

b Because the alternating harmonic series does not converge absolutely, a rearrangement of the terms in the series will result in series whose sums vary. One such rearrangement in alternating p positive terms and n negative terms leads to the following sum[1]:

[1] This is discussed by Lawrence H. Riddle in the *Kenyon Math. Quarterly*, 1(2), 6-21.

$$\begin{aligned} \frac{1}{2}\ln\frac{4p}{n} &= \underbrace{\left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)}_{p \text{ terms}}-\underbrace{\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)}_{n \text{ terms}} \\ &\quad+\underbrace{\left(\frac{1}{2p+1}+\cdots+\frac{1}{4p-1}\right)}_{p \text{ terms}}-\underbrace{\left(\frac{1}{2n+2}+\cdots+\frac{1}{4n}\right)}_{n \text{ terms}}+\cdots. \end{aligned}$$

Find rearrangements of the alternating harmonic series to give the following sums; that is, determine p and n for the given expression and write down the above series explicitly; that is, determine p and n leading to the following sums.

Note: This problem does not have unique solutions.

i. $\frac{5}{2}\ln 2$.
In each case we need to rewrite the given expression in the form $\frac{1}{2}\ln\frac{4p}{n}$ and select values for n and p. In the first problem we have

$$\begin{aligned}\frac{1}{2}\ln\frac{4p}{n} &= \frac{5}{2}\ln 2\\ &= \frac{1}{2}\ln 2^5 = \frac{1}{2}\ln 32.\end{aligned}$$

For this problem we can pick $n = 1$ and $p = 8$. Then, we would have

$$\begin{aligned}\frac{5}{2}\ln 2 &= \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}\right)-\frac{1}{2}\\ &\quad+\left(\frac{1}{17}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}+\frac{1}{25}+\frac{1}{27}+\frac{1}{29}+\frac{1}{31}\right)-\frac{1}{4}+\ldots.\end{aligned}$$

ii. $\ln 8$.
In this problem we have

$$\begin{aligned}\frac{1}{2}\ln\frac{4p}{n} &= \ln 8\\ &= \frac{1}{2}\ln 64.\end{aligned}$$

For this problem we can pick $n = 1$ and $p = 16$. The rearranged series is then

$$\begin{aligned}\ln 8 &= \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}\right.\\ &\quad\left.+\frac{1}{17}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}+\frac{1}{25}+\frac{1}{27}+\frac{1}{29}+\frac{1}{31}\right)-\frac{1}{2}+\ldots.\end{aligned}$$

iii. 0.
In this problem we have $\frac{1}{2}\ln\frac{4p}{n} = 0$. Therefore, $n = 4p$. We can pick $p = 1$ and $n = 4$. The rearranged series is then

$$\begin{aligned}0 &= 1-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)+\frac{1}{3}-\left(\frac{1}{10}+\frac{1}{12}+\frac{1}{14}+\frac{1}{16}\right).\\ &\quad+\frac{1}{5}-\left(\frac{1}{18}+\frac{1}{20}+\frac{1}{22}+\frac{1}{24}\right)+\frac{1}{7}-\ldots.\end{aligned}$$

iv. A sum that is close to π.
For this problem we have $\pi \approx \frac{1}{2}\ln\frac{4p}{n}$, or $\frac{4p}{n} \approx e^{2\pi} \approx 535.4916560$. So, one choice is $n = 1$ and $p = 535.49/4 \approx 134$. Thus, there is one positive term followed by 134 negative terms, etc.

8. Determine the radius and interval of convergence of the following infinite series:

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^n}{n}$.
Using the nth Root Test, we have

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-1|}{\sqrt[n]{n}} = |x-1| < 1.$$