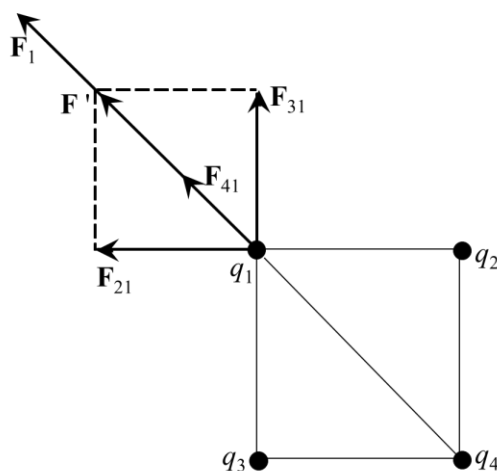


## Chapter 1

**Problem 1.1** Four identical point charges  $q$  are placed at the corners of a square with side  $a$ . Find the net force (magnitude and direction) acting on each of the charges, and the potential energy of the system. The charges are located in vacuum.



**Figure 1S.1** Forces acting on charge  $q_1$  from charges  $q_2$ ,  $q_3$ , and  $q_4$ .

**Solution** Figure 1S.1 illustrates the forces exerted by charges  $q_2$ ,  $q_3$ , and  $q_4$  on charge  $q_1$ . Since the charges are equal to each other and two of them ( $q_2$  and  $q_3$ ) are at the same distance from charge  $q_1$ , it follows that the magnitude of forces  $F_{21}$  and  $F_{31}$  are equal to each other:

$$F_{21} = F_{31} = \frac{q_1 q_2}{4\pi\epsilon_0 a^2}.$$

The distance between the charges  $q_1$  and  $q_4$  is equal to  $a\sqrt{2}$ . The magnitude of force  $F_{41}$  is equal to:

$$F_{41} = \frac{q_1 q_4}{4\pi\epsilon_0 2a^2}.$$

In order to find the resultant of all the forces it is necessary to add vectors:

$$\mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}_{41} = \mathbf{F}_1$$

Let us denote:  $\mathbf{F}_{21} + \mathbf{F}_{31} = \mathbf{F}'$ . The magnitude of force  $\mathbf{F}'$  is equal to:

$$F' = \sqrt{(F_{21})^2 + (F_{31})^2} = \sqrt{2}F_{21},$$

$$F' = \frac{\sqrt{2}q_1q_2}{4\pi\epsilon_0a^2}.$$

Forces  $\mathbf{F}'$  and  $\mathbf{F}_{41}$  act along the same direction; thus  $F_1$  is equal to:

$$F_1 = F' + F_{41},$$

$$F_1 = \frac{\sqrt{2}q_2q_1}{4\pi\epsilon_0a^2} + \frac{q_1q_4}{4\pi\epsilon_02a^2} = \frac{q^2}{4\pi\epsilon_0a^2}(\sqrt{2} + \frac{1}{2}),$$

$$F_1 = \frac{q^2}{8\pi\epsilon_0a^2}(2\sqrt{2} + 1).$$

The resultant force  $\mathbf{F}_1$  is directed along the diagonal of the square. Similarly, one can find the forces acting on each of the charges  $q_2$ ,  $q_3$  and  $q_4$ . It is easy to show that the magnitude of forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ ,  $\mathbf{F}_3$ , and  $\mathbf{F}_4$  are equal to each other.

The potential energy of the system of charges is the sum:

$$U = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}.$$

Here  $U_{ij}$  is the interaction energy between charges  $q_i$  and  $q_j$ . Since the charges are point-like, then:

$$U_{12} = U_{23} = U_{34} = U_{14} = \frac{q^2}{4\pi\epsilon_0a},$$

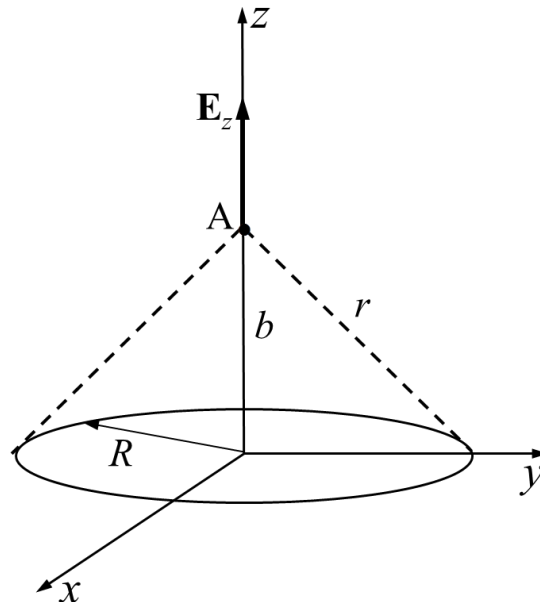
$$U_{13} = U_{24} = \frac{q^2}{4\pi\epsilon_0a\sqrt{2}}$$

$$U = \frac{4q^2}{4\pi\epsilon_0a} + \frac{2q^2}{4\pi\epsilon_0a\sqrt{2}} = \frac{1}{4\pi\epsilon_0} \times \frac{q^2}{a}(4 + \sqrt{2}).$$

**Problem 1.2** Charge  $q$  is uniformly distributed on a very thin ring of radius  $R$  placed in vacuum. Find the electric potential and the electric field at a point on the normal to the ring plane that passes through the ring center (Fig. 1S.2) as a function of the distance  $b$  from the center of the ring.

**Solution** First we will determine the potential using the principle of superposition. Then, using the equation  $\mathbf{E} = -\text{grad}\varphi$  we will obtain an expression for the electric field. The electric potential of the resulting field at an arbitrary point A on the axis of the ring can be found by integrating the contributions of all the ring:

$$\varphi = \oint_{(L)} d\varphi .$$



**Figure 1S.2** Electric field of a circular charged ring of radius  $R$ .

The charge  $dq$  of a ring segment of length  $dl$  is equal to:

$$dq = \lambda dl = \frac{q}{2\pi R} dl.$$

The potential generated by this charge at a point A, which lies at a distance  $b$  from the center of the ring, is equal to:

$$d\varphi = \frac{dq}{4\pi\epsilon_0 r},$$

where  $r$  is the distance from the segment  $dl$  to point A. Using these relations and the fact that  $r = \sqrt{R^2 + b^2}$ , we get:

$$\varphi = \frac{q}{2\pi R} \cdot \frac{1}{4\pi\epsilon_0 \sqrt{R^2 + b^2}} \int_0^{2\pi R} dl = \frac{q}{4\pi\epsilon_0 \sqrt{R^2 + b^2}}.$$

In order to calculate the electric field  $\mathbf{E}$  at point A, we find the projections  $E_x$ ,  $E_y$  and  $E_z$  of the electric field on the coordinate axes. Because of the symmetry of the charge distribution  $E_x = 0$ ,  $E_y = 0$ .

Taking into account that in our case  $z = b$  and using the expression for the potential, we get:

$$E_z(b) = -\left. \frac{d\varphi}{dz} \right|_{z=b} = -\left. \frac{d}{dz} \left( \frac{q}{4\pi\epsilon_0 \sqrt{R^2 + z^2}} \right) \right|_{z=b} = \left. \frac{qz}{4\pi\epsilon_0 \sqrt{(R^2 + z^2)^3}} \right|_{z=b}.$$

Substituting  $z = b$ , we obtain:

$$E_z = \frac{qb}{4\pi\epsilon_0 \sqrt{(R^2 + b^2)^3}}.$$

Thus, for the potential and electric field at point A on the axis of the ring, we obtain:

$$\varphi = \frac{k_e q}{\sqrt{R^2 + b^2}}, \quad E_z = \frac{k_e qb}{(R^2 + b^2)^{3/2}}.$$

**Problem 1.3** A non-polar molecule is located on the normal to a ring plane (radius  $R$ ) that passes through the ring center, at a distance  $z$  from the ring center. The non-polar molecule dipole moment is proportional to the electric field at the location of the molecule, i.e.  $\mathbf{p} = \epsilon_0 \alpha \mathbf{E}$ , where  $\alpha$  is the polarizability of the molecule. Determine the distance  $z$  from the center of the ring, where the force  $\mathbf{F}$  acting on the molecule is equal to zero. Assume that the system is in vacuum and ignore all other forces acting on the dipole. Note: there are two such positions. Hint: see Exercise 1.2.

**Solution** Solving the problem 1.2 we obtained the following expression for the electric field:

$$E = \frac{qz}{4\pi\epsilon_0(R^2 + z^2)^{3/2}}.$$

The potential energy of a dipole in an electric field  $\mathbf{E}$  is determined by the equation:  $U = -(\mathbf{p} \cdot \mathbf{E}) = -\epsilon_0 \alpha E^2$ . The force acting on the dipole is related to the potential energy through the relation:

$$\mathbf{F} = -\text{grad}U = -\nabla U.$$

The force component along the  $z$ -axis is equal to:

$$F_z = -\frac{dU}{dz} = -\frac{d}{dz}(-\epsilon_0 \alpha E^2) = 2\epsilon_0 \alpha E \frac{dE}{dz}.$$

The component  $F_z = 0$ , either if  $E = 0$  or if  $\frac{dE}{dz} = 0$ . In the first case  $z_1 = 0$  and in the

second case  $\frac{d}{dz} \left( \frac{z}{(R^2 + z^2)^{3/2}} \right) = 0$ . Thus, we find  $z_2 = \pm R / \sqrt{2}$ .

**Problem 1.4** Two metal spheres of radii  $R_1$  and  $R_2$  have charges  $Q_1$  and  $Q_2$  and they are located in vacuum. Find the energy  $U$  that will be released if the spheres are connected by a thin conductor.

**Solution** According to the law of conservation of energy, the energy released, when the spheres are connected, is equal to the energy difference between the spheres before and after the connection, i.e.  $U = U_1 - U_2$  where  $U_1$  is the energy of spheres before and  $U_2$  is the energy of the spheres after the connection is made. Therefore, the problem can be solved by finding the energy of spheres before and after connection. Since we know the charge of spheres before the connection, the energy of the spheres is given by the sum of the energies of each sphere:

$$U_1 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2},$$

where  $C_i$  is the capacitance of sphere  $i$ . After the connection is made the spheres can be represented by two capacitors connected in parallel with an equivalent capacitance  $C = C_1 + C_2$ .

According to the law of conservation of charge the charge of the spheres after their connection is equal to:

$$Q = Q_1 + Q_2.$$

The energy of spheres after the connection is made is equal to:

$$U_2 = \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}.$$

The energy released is equal to:

$$U = U_1 - U_2 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} - \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}.$$

The capacitances of spheres are equal to:  $C_1 = 4\pi\epsilon_0 R_1$  and  $C_2 = 4\pi\epsilon_0 R_2$ .

Substituting  $C_1$  and  $C_2$  in the above relations, we get:

$$U = \frac{1}{2} \frac{Q_1^2}{4\pi\epsilon_0 R_1} + \frac{1}{2} \frac{Q_2^2}{4\pi\epsilon_0 R_2} - \frac{(Q_1 + Q_2)^2}{4\pi\epsilon_0 (R_1 + R_2)}.$$

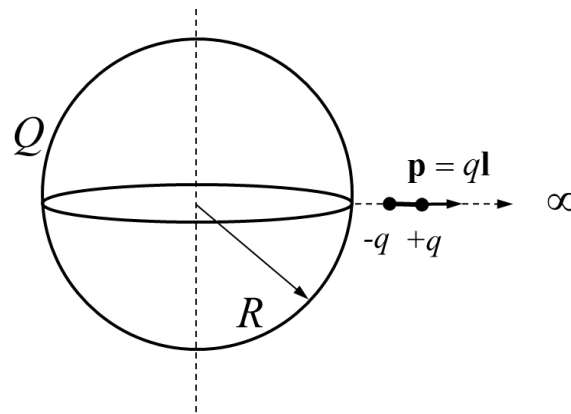
From the above equation we obtain:

$$U = \frac{1}{8\pi\epsilon_0} \frac{(R_2 Q_1 - R_1 Q_2)^2}{R_1 R_2 (R_1 + R_2)}.$$

Note: the energy released depends strongly on the sign and the absolute value of the charges on the spheres. Opposite signs of initial charges leads to higher energy then for the case of two initial charges with the same sign. The energy is zero if  $R_2 Q_1 = R_1 Q_2$ .

**Problem 1.5** Find the work that must be done by an external agent to move a small electric dipole of dipole moment  $p$  from the surface of a uniformly charged sphere to infinity. The

sphere radius is equal to  $R$  and its charge equal to  $Q$ . The dipole moment is oriented radially (see Fig. 1S.3).



**Figure 1S.3** Interaction of a charged sphere with an electric dipole.

**Solution** The electric field outside a uniformly charged sphere in the space is the same as the field of a point charge placed at the center of the sphere:

$$\mathbf{E} = \frac{Q\mathbf{r}}{4\pi\epsilon_0 r^3}$$

Electric dipole has a dipole moment  $\mathbf{p} = q\mathbf{l}$  where  $l$  is a distance between the charges  $-q$  and  $+q$ . For simplicity vector  $\mathbf{l}$  (and hence vector  $\mathbf{p}$ ) are chosen to be parallel to vector  $\mathbf{r}$  (see Fig. 1S.3). If during the move the relative position of charges  $-q$  and  $+q$  remains unchanged, the work to move the dipole can be calculated as the work  $W_-$  and  $W_+$  be done to move separately charges  $-q$  and  $+q$ . In accordance with Eq. 1.24 these works are:

$$W_{\mp} = \mp \frac{k_e q Q}{r_{1,2}}, \text{ where } r_1 = R, r_2 = R + l.$$

Thus, work done to move the dipole from the surface of the sphere to infinity is equal to:

$$W = W_+ + W_- = k_e q Q \left( \frac{1}{r_2} - \frac{1}{r_1} \right) = k_e q Q \frac{r_1 - r_2}{r_1 r_2} = -\frac{Qp}{4\pi\epsilon_0 R^2} \quad (l \ll R \text{ and } k_e = \frac{1}{4\pi\epsilon_0}).$$

The same result can be obtained using equation for a potential energy  $U = -\mathbf{p}\mathbf{E}$  of a dipole in the electric field  $\mathbf{E}$  as its potential energy at the sphere is  $U_1 = -\frac{pQ}{4\pi\epsilon_0 R^2}$  and at the infinity

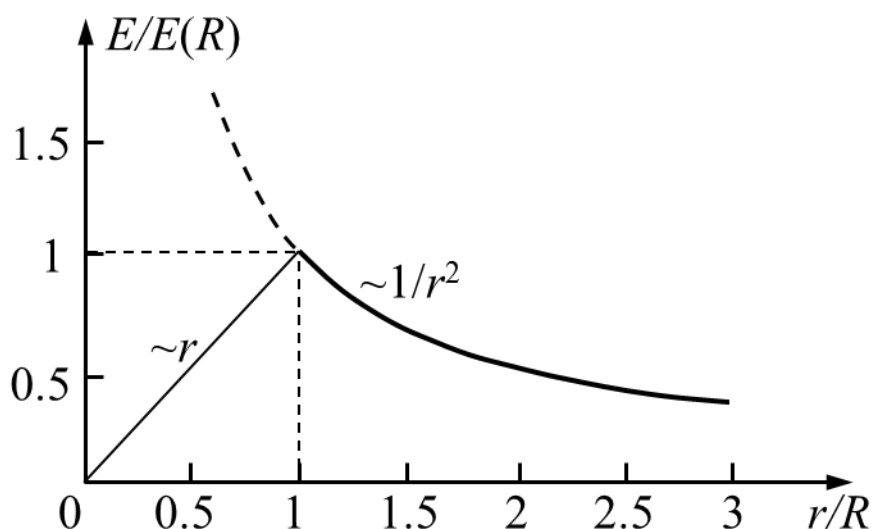
$U_2 = 0$ , so the work  $W = -(U_2 - U_1)$  is the same as written above.

**Problem 1.6** Consider an insulating sphere with a radius  $R$ , uniformly charged over its volume with a total charge  $Q$ . Determine the energy of the electric field inside and outside the sphere, if the electric field in the sphere and outside of the sphere is:

$$E(r) = \begin{cases} \frac{k_e Q}{r^2}, & r \geq R, \\ \frac{k_e Q}{R^2} \cdot \frac{r}{R}, & r \leq R. \end{cases}$$

**Solution.** Outside the sphere, the field is equal to the field of the point charge  $Q$  placed at the sphere center as shown above. Inside the sphere, the field at a point located at a distance  $r$  from its center is the field generated by a point charge  $Q^{(in)} = Qr^3 / R^3$  placed at the sphere center and equal to the charge inside the sphere with the radius  $r$  (the volume of the whole sphere is  $4\pi R^3 / 3$  and the volume of sphere of radius  $r$  that contributes to the potential is  $4\pi r^3 / 3$ ). Fig. 1S.4 shows distribution of the electric field given by the above relation. Let us introduce this expression into the Eq. (1.34) for the energy density of the electric field and take into account Eq. (1.4) of the text:

$$u_e(r) = \frac{k_e Q^2}{8\pi} \begin{cases} \frac{1}{r^4}, & r \geq R, \\ \frac{r^2}{R^6}, & r \leq R. \end{cases}$$



**Figure 1S.4** Electric field of a uniformly charged dielectric sphere.



We will calculate the electric field energy  $U_1$  outside the insulating sphere and the energy  $U_2$  inside the sphere using Eqs. (1.35) of the text. Given the spherical symmetry of the problem, the volume element  $d(Vol) = 4\pi r^2 dr$ . The following relations give these energies:

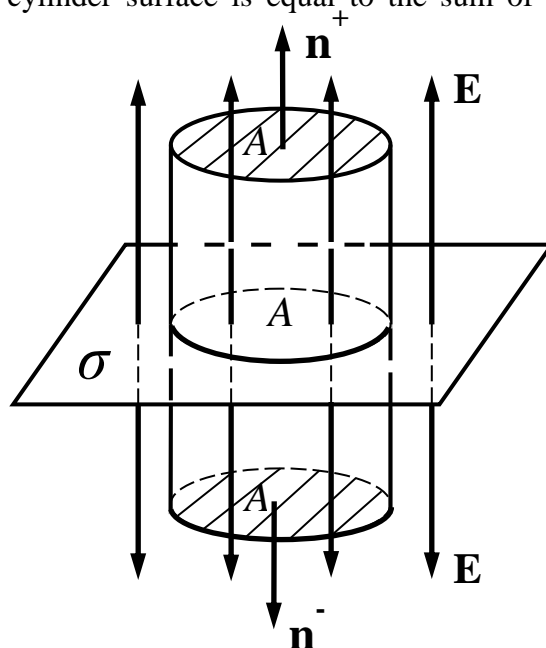
$$U_1 = \frac{k_e Q^2}{8\pi} \int_R^\infty \frac{4\pi r^2}{r^4} dr = \frac{k_e Q^2}{2R},$$

$$U_2 = \frac{k_e Q^2}{8\pi} \int_0^R \frac{4\pi r^2 \cdot r^2}{R^6} dr = \frac{k_e Q^2}{10R}.$$

The ratio  $U_1 / U_2$  of the energies outside and inside the sphere does not depend on the sphere radius and is equal to 5.

**Problem 1.7** Using the integral form of Gauss's law, find the electric field in the vicinity of an infinite uniformly charged insulating plane with constant surface charge density  $\sigma$ .

**Solution** Let the Gaussian surface be a cylinder with a base area  $A$  (Fig. 1S.5). The cylinder bases are perpendicular to the electric field lines on both sides of the plane. The total electric field flux through the cylinder surface is equal to the sum of fluxes through both bases, i.e.  $\Phi_E = 2EA$ .



**Figure 1S.5** Electric field in the vicinity of a charged surface.

A charge enclosed inside the cylinder is  $q = \sigma A$ . According to Eq. (1.42)  $\Phi_E = q / \epsilon_0 = \sigma A / \epsilon_0$ . Therefore, comparing two different expressions for  $\Phi_E$  we conclude that  $2EA = \sigma A / \epsilon_0$ . Thus, the magnitude of the electric field of an infinite uniformly charged insulating surface at all points outside the surface is given by the equation:  $E = \sigma / 2\epsilon_0$ .

**Problem 1.8** The potential associated with an electric field depends on the Cartesian coordinates according the equations: (i)  $\varphi = a(x^2 - y^2)$  and (ii)  $\varphi = bxyz$  where  $a$  and  $b$  are constants. Determine the electric field in each case.

**Solution** The components of the electric field are given by the equations:

$$E_x = -\frac{\partial \varphi}{\partial x}, \quad E_y = -\frac{\partial \varphi}{\partial y}, \quad E_z = -\frac{\partial \varphi}{\partial z}.$$

(i) In the first case, we have  $E_x = -2ax$ ,  $E_y = 2ay$ ,  $E_z = 0$ . Therefore,

$$\mathbf{E} = 2a(-x\mathbf{i} + y\mathbf{j}).$$

(ii) In the second case we get:  $E_x = -byz$ ,  $E_y = -bxz$ ,  $E_z = -bxy$ . Thus,

$$\mathbf{E} = -b(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}).$$

**Problem 1.9** Determine the electric field  $\mathbf{E}$  of an electric dipole with a dipole moment  $\mathbf{p}$  at a point located at a distance  $r_-$  and  $r_+$  from the charges  $-q$  and  $+q$  as shown in Fig. 1S.6 (here  $q > 0$ ). Assume that  $r_-$  and  $r_+$  are much larger than the charge separation  $l$ .

**Solution** In the textbook, we obtained the following expression for the dipole potential (Eq. (1.28)):

$$\varphi_{dip} = k_e \frac{ql \cos \alpha}{r^2} = k_e \frac{q\mathbf{l} \cdot \mathbf{r}}{r^3} = k_e ql \frac{x}{(x^2 + y^2)^{3/2}}.$$

According to Eq. (1.49), in the two dimensional coordinate system we can write

$$\mathbf{E} = -\text{grad } \varphi = -\left(\frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j}\right),$$

where

$$\frac{\partial \varphi}{\partial x} = k_e q l \frac{\partial}{\partial x} \left[ \frac{x}{(x^2 + y^2)^{3/2}} \right] = k_e q l \left[ \frac{(x^2 + y^2)^{3/2} - 3x^2(x^2 + y^2)^{1/2}}{(x^2 + y^2)^3} \right] = \frac{k_e q l}{(x^2 + y^2)^{5/2}} (y^2 - 2x^2),$$

$$\frac{\partial \varphi}{\partial y} = k_e q l \frac{\partial}{\partial y} \left[ \frac{x}{(x^2 + y^2)^{3/2}} \right] = k_e q l x \left( -\frac{3}{2} \right) \frac{2y}{(x^2 + y^2)^{5/2}} = \frac{k_e q l}{(x^2 + y^2)^{5/2}} (-3xy).$$

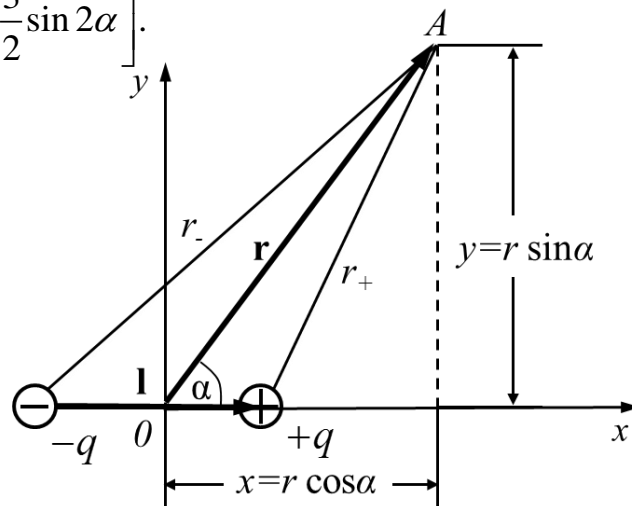
As shown in Fig. 1S.6, we have  $x = r \cos \alpha$ ,  $y = r \sin \alpha$  and  $r = (x^2 + y^2)^{1/2}$ ,

$$\frac{\partial \varphi}{\partial x} = \frac{k_e q l}{r^5} (r^2 \sin^2 \alpha - 2r^2 \cos^2 \alpha) = \frac{k_e q l}{r^3} (\sin^2 \alpha - 2 \cos^2 \alpha),$$

$$\frac{\partial \varphi}{\partial y} = \frac{k_e q l}{r^5} (-3r \cos \alpha \cdot r \sin \alpha) = \frac{k_e q l}{r^3} (-3 \sin \alpha \cos \alpha).$$

Thus, the electric field  $\mathbf{E}$  is

$$\begin{aligned} \mathbf{E} &= -\left(\frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j}\right) = -\frac{k_e q l}{r^3} [(\sin^2 \alpha - 2 \cos^2 \alpha) \mathbf{i} - (3 \sin \alpha \cos \alpha) \mathbf{j}] = \\ &= \frac{k_e q l}{r^3} \left[ \mathbf{i} (3 \cos^2 \alpha - 1) + \mathbf{j} \frac{3}{2} \sin 2\alpha \right]. \end{aligned}$$



**Figure 1S.6** Electric potential of a dipole at a point  $A$ .

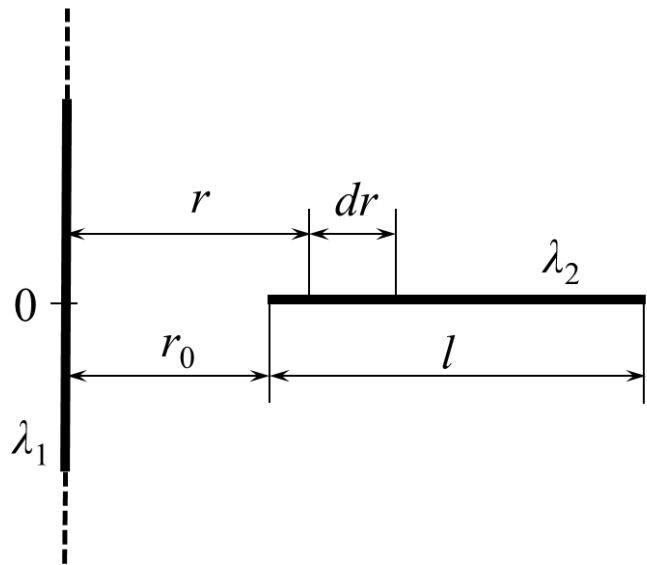
**Problem 1.10**

An infinitely long insulating rod is uniformly charged with linear charge density  $\lambda_1 = 3.00 \times 10^{-7} \text{ C/m}$ . A second insulating rod of finite length  $l = 20.0 \text{ cm}$  is uniformly charged with linear density  $\lambda_2 = 2.00 \times 10^{-7} \text{ C/m}$ . The two rods have their axes perpendicular to each other as shown in Fig. 1S.7. The distance  $r_0 = 10.0 \text{ cm}$ . Determine the electric force between the two rods.

**Solution.** In order to find the force  $F$  we use the relationship

$$d\mathbf{F} = \mathbf{E}dq \text{ .}$$

We assume that the first rod creates around it an electric field, where the second rod is placed. If we consider a segment (element) of length  $dr$  on the second rod its charge  $dq = \lambda_2 dr$  can be considered as a point charge (see Fig. 1S.7.)



**Figure 1S.7** Interaction of an infinitely long insulating rod with charge density  $\lambda_1$  and an insulating rod of length  $l$  with linear density  $\lambda_2$ .

Consider the force  $d\mathbf{F}$  exerted by the electric field  $\mathbf{E}$  of the first rod on charge  $dq$ . The electric field at the location of the electric charge  $dq$  is equal to: