

## Solution Manual

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### Section 1.1

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|-----------------------------|-----------------------------|
| 1. first-order, linear      | 2. first-order, nonlinear   |
| 3. first-order, nonlinear   | 4. third-order, linear      |
| 5. second-order, linear     | 6. first-order, nonlinear   |
| 7. third-order, nonlinear   | 8. second-order, linear     |
| 9. second-order, nonlinear  | 10. first-order, nonlinear  |
| 11. first-order, nonlinear  | 12. second-order, nonlinear |
| 13. first-order, nonlinear  | 14. third-order, linear     |
| 15. second-order, nonlinear | 16. third-order, nonlinear  |

### Section 1.2

1. Because the differential equation can be rewritten  $e^{-y} dy = x dx$ , integration immediately gives  $-e^{-y} = \frac{1}{2}x^2 - C$ , or  $y = -\ln(C - x^2/2)$ .
2. Separating variables, we have that  $dx/(1+x^2) = dy/(1+y^2)$ . Integrating this equation, we find that  $\tan^{-1}(x) - \tan^{-1}(y) = \tan(C)$ , or  $(x-y)/(1+xy) = C$ .
3. Because the differential equation can be rewritten  $\ln(x)dx/x = y dy$ , integration immediately gives  $\frac{1}{2}\ln^2(x) + C = \frac{1}{2}y^2$ , or  $y^2(x) - \ln^2(x) = 2C$ .
4. Because the differential equation can be rewritten  $y^2 dy = (x + x^3) dx$ , integration immediately gives  $y^3(x)/3 = x^2/2 + x^4/4 + C$ .
5. Because the differential equation can be rewritten  $y dy/(2+y^2) = x dx/(1+x^2)$ , integration immediately gives  $\frac{1}{2}\ln(2+y^2) = \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(C)$ , or  $2+y^2(x) = C(1+x^2)$ .
6. Because the differential equation can be rewritten  $dy/y^{1/3} = x^{1/3} dx$ , integration immediately gives  $\frac{3}{2}y^{2/3} = \frac{3}{4}x^{4/3} + \frac{3}{2}C$ , or  $y(x) = \left(\frac{1}{2}x^{4/3} + C\right)^{3/2}$ .

7. Because the differential equation can be rewritten  $e^{-y} dy = e^x dx$ , integration immediately gives  $-e^{-y} = e^x - C$ , or  $y(x) = -\ln(C - e^x)$ .

8. Because the differential equation can be rewritten  $dy/(y^2 + 1) = (x^3 + 5) dx$ , integration immediately gives  $\tan^{-1}(y) = \frac{1}{4}x^4 + 5x + C$ , or  $y(x) = \tan(\frac{1}{4}x^4 + 5x + C)$ .

9. Because the differential equation can be rewritten  $y^2 dy/(b - ay^3) = dt$ , integration immediately gives  $\ln[b - ay^3] \Big|_{y_0}^y = -3at$ , or  $(ay^3 - b)/(ay_0^3 - b) = e^{-3at}$ .

10. Because the differential equation can be written  $du/u = dx/x^2$ , integration immediately gives  $u = Ce^{-1/x}$  or  $y(x) = x + Ce^{-1/x}$ .

11. From the hydrostatic equation and ideal gas law,  $dp/p = -g dz/(RT)$ . Substituting for  $T(z)$ ,

$$\frac{dp}{p} = -\frac{g}{R(T_0 - \Gamma z)} dz.$$

Integrating from 0 to  $z$ ,

$$\ln \left[ \frac{p(z)}{p_0} \right] = \frac{g}{R\Gamma} \ln \left( \frac{T_0 - \Gamma z}{T_0} \right), \quad \text{or} \quad \frac{p(z)}{p_0} = \left( \frac{T_0 - \Gamma z}{T_0} \right)^{g/(R\Gamma)}.$$

12. For  $0 < z < H$ , we simply use the previous problem. At  $z = H$ , the pressure is

$$p(H) = p_0 \left( \frac{T_0 - \Gamma H}{T_0} \right)^{g/(R\Gamma)}.$$

Then we follow the example in the text for an isothermal atmosphere for  $z \geq H$ .

13. Separating variables, we find that

$$\frac{dV}{V + RV^2/S} = \frac{dV}{V} - \frac{R dV}{S(1 + RV/S)} = -\frac{dt}{RC}.$$

Integration yields

$$\ln \left( \frac{V}{1 + RV/S} \right) = -\frac{t}{RC} + \ln(C).$$

Upon applying the initial conditions,

$$V(t) = \frac{V_0}{1 + RV_0/S} e^{-t/(RC)} + \frac{RV_0/S}{1 + RV_0/S} e^{-t/(RC)} V(t).$$

Solving for  $V(t)$ , we obtain

$$V(t) = \frac{SV_0 e^{-t/(RC)}}{S + RV_0 [1 - e^{-t/(RC)}]}.$$

14. From the definition of  $\gamma$ , we can write the differential equation

$$\frac{A}{B} \frac{dT}{dt} + T^4 = \gamma^4,$$

or

$$\begin{aligned} \frac{B}{A} dt &= -\frac{dT}{T^4 - \gamma^4} = \frac{1}{2\gamma^2} \left[ \frac{dT}{T^2 + \gamma^2} - \frac{dT}{T^2 - \gamma^2} \right] \\ &= \frac{1}{4\gamma^3} \left[ \frac{2\gamma dT}{T^2 + \gamma^2} - \frac{dT}{T - \gamma} + \frac{dT}{T + \gamma} \right]. \end{aligned}$$

The final answer follows from direction integration.

15. Separating the variables yields

$$\frac{dN}{N \ln(K/N)} = b dt, \quad \text{or} \quad \frac{d[\ln(K/N)]}{\ln(K/N)} = -b dt.$$

Integration leads to

$$\ln [\ln(K/N)] - \ln \{\ln[K/N(0)]\} = -bt$$

or

$$\ln \{\ln(K/N)/\ln[K/N(0)]\} = -bt$$

or

$$\ln(K/N) = \ln[K/N(0)]e^{-bt}$$

or

$$\ln[N/N(0)] = \ln[K/N(0)] (1 - e^{-bt})$$

or

$$N(t) = N(0) \exp\{\ln[K/N(0)] (1 - e^{-bt})\}.$$

16. Separating the variables yields

$$\frac{dI}{I} - \frac{\beta}{\alpha} \frac{dI}{1 + \beta I/\alpha} = -\alpha dz.$$

Integration leads to

$$\ln \left[ \frac{I(z)}{1 + \beta I(z)/\alpha} \frac{1 + \beta I(0)/\alpha}{I(0)} \right] = -\alpha z,$$

or

$$\frac{I(z)}{1 + \beta I(z)/\alpha} = \frac{I(0)}{1 + \beta I(0)/\alpha} e^{-\alpha z}, \quad \text{or} \quad I(z) = \frac{\alpha I(0) e^{-\alpha z}}{\alpha + \beta I(0) [1 - e^{-\alpha z}]}.$$

17. Separating the variables yields

$$\begin{aligned} & \frac{d[X]}{([A]_0 - [X])([B]_0 - [X])([C]_0 - [X])} = k dt \\ & \frac{d[X]}{([A]_0 - [B]_0)([A]_0 - [C]_0)([A]_0 - [X])} \\ & + \frac{d[X]}{([B]_0 - [A]_0)([B]_0 - [C]_0)([B]_0 - [X])} \\ & + \frac{d[X]}{([C]_0 - [A]_0)([C]_0 - [B]_0)([C]_0 - [X])} = k dt \end{aligned}$$

Integration yields

$$\begin{aligned} & \frac{1}{([A]_0 - [B]_0)([A]_0 - [C]_0)} \ln\left(\frac{[A]_0}{[A]_0 - [X]}\right) \\ & + \frac{1}{([B]_0 - [A]_0)([B]_0 - [C]_0)} \ln\left(\frac{[B]_0}{[B]_0 - [X]}\right) \\ & + \frac{1}{([C]_0 - [A]_0)([C]_0 - [B]_0)} \ln\left(\frac{[C]_0}{[C]_0 - [X]}\right) = kt. \end{aligned}$$

18. Separation of variables yields

$$\frac{d[X]}{\alpha - [X]} = (k_1 + k_2) dt.$$

Integrating both sides,

$$\ln(\alpha - [X]) - \ln(\alpha - [X]_0) = -(k_1 + k_2)t.$$

Because  $[X]_0 = 0$ ,

$$\alpha - [X] = \alpha e^{-(k_1 + k_2)t}, \quad \text{or} \quad [X] = \alpha \left[1 - e^{-(k_1 + k_2)t}\right].$$

### Section 1.3

1. Because  $M(x, y) = -y$  and  $N(x, y) = x + y$ , we have that  $M(tx, ty) = -ty = tM(x, y)$ , and  $N(tx, ty) = tx + ty = tN(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,  $(ux + x)(u dx + x du) = ux dx$ , or  $-u^2 x dx = (1 + u)x^2 du$ , or

$$-\frac{dx}{x} = \left( \frac{1}{u} + \frac{1}{u^2} \right) du.$$

Integrating this last equation,

$$-\ln|x| = \ln(u) - \frac{1}{u} - C, \quad \text{or} \quad \ln|y| - \frac{x}{y} = C.$$

2. Because  $M(x, y) = y - x$  and  $N(x, y) = x + y$ , we have that  $M(tx, ty) = ty - tx = tM(x, y)$ , and  $N(tx, ty) = tx + ty = tN(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,  $(u - 1)x dx + (u + 1)x(u dx + x du) = 0$ , or

$$(u^2 + 2u - 1) dx = -(u + 1)x du, \quad \text{or} \quad -\frac{dx}{x} = \frac{u + 1}{u^2 + 2u - 1} du.$$

Integrating this last equation,

$$-\ln|x| = \frac{1}{2} \ln|u^2 + 2u - 1| + C, \quad \text{or} \quad x^2 \left( \frac{y^2}{x^2} + 2\frac{y}{x} - 1 \right) = y^2 + 2xy - x^2 = C.$$

3. Because  $M(x, y) = x^2 + y^2$  and  $N(x, y) = 2xy$ , we have that  $M(tx, ty) = t^2 x^2 + t^2 y^2 = t^2(x^2 + y^2) = t^2 M(x, y)$ , and  $N(tx, ty) = 2t^2 xy = t^2 N(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,

$$2x(ux)(u dx + x du) + (x^2 + x^2 u^2) dx = 0$$

or

$$2xu du + (1 + 3u^2) dx = 0, \quad \text{or} \quad \frac{dx}{x} = -\frac{2u}{1 + 3u^2} du.$$

Integrating this last equation,

$$\ln|x| = -\frac{1}{3} \ln(1 + 3u^2) + \ln(C_1).$$

Inverting the logarithms,

$$|x|(1 + 3y^2/x^2)^{1/3} = C_1, \quad \text{or} \quad |x|(x^2 + 3y^2) = C.$$

4. Because  $M(x, y) = y(y - x)$  and  $N(x, y) = x(x + y)$ , we have that  $M(tx, ty) = ty(ty - tx) = t^2 M(x, y)$ , and  $N(tx, ty) = tx(tx + ty) = t^2 N(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,

$$x^2 u(u-1) dx + x^2(u+1)(u dx + x du) = 0$$

or

$$2u^2 dx + (u+1)x du = 0, \quad \text{or} \quad 2 \frac{dx}{x} = -\frac{u+1}{u^2} du.$$

Integrating this last equation,

$$\ln|x|^2 = -\ln|u| + \frac{1}{u^2} + C, \quad \text{or} \quad \ln|ux^2| = C - \frac{x}{y}, \quad \text{or} \quad \ln|xy| = C - \frac{x}{y}.$$

5. Because  $M(x, y) = y + 2\sqrt{xy}$  and  $N(x, y) = -x$ , we have that  $M(tx, ty) = ty + 2\sqrt{t^2xy} = ty + 2t\sqrt{xy} = tM(x, y)$ , and  $N(tx, ty) = -tx = tN(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,

$$x(u dx + x du) = (xu + 2x\sqrt{u}) dx, \quad \text{or} \quad \frac{du}{2\sqrt{u}} = \frac{dx}{x}.$$

Integrating this last equation,

$$u^{1/2} = \ln|x| + C, \quad \text{or} \quad y = x(\ln|x| + C)^2.$$

6. Because  $M(x, y) = \sqrt{x^2 + y^2} - y$  and  $N(x, y) = x$ , we have that  $M(tx, ty) = \sqrt{t^2x^2 + t^2y^2} - ty = t(\sqrt{x^2 + y^2} - y) = tM(x, y)$ , and  $N(tx, ty) = tx = tN(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,

$$\left(\sqrt{x^2 + x^2u^2} - ux\right) dx + x(x du + u dx) = 0,$$

or

$$x\sqrt{1+u^2} dx + x^2 du = 0, \quad \text{or} \quad \frac{dx}{x} = -\frac{du}{\sqrt{1+u^2}}.$$

Integrating this last equation,

$$-\ln(x) = -\ln\left(u + \sqrt{1+u^2}\right) - \ln(C).$$

Inverting the logarithms,

$$ux + \sqrt{u^2x^2 + x^2} = C, \quad \text{or} \quad y + \sqrt{x^2 + y^2} = C.$$

7. Because  $M(x, y) = \sec(y/x) + y/x$  and  $N(x, y) = -1$ , we have that  $M(tx, ty) = \sec[(ty)/(tx)] + (ty)/(tx) = \sec(y/x) + y/x = M(x, y)$ , and

$N(tx, ty) = -1 = N(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,

$$u dx + x du = [\sec(u) + u] dx, \quad \text{or} \quad \cos(u) du = \frac{dx}{x}.$$

Integrating and substituting for  $u$ , the final answer is

$$\sin(y/x) - \ln|x| = C.$$

8. Because  $M(x, y) = e^{y/x} + y/x$  and  $N(x, y) = -1$ , we have that  $M(tx, ty) = e^{(ty)/(tx)} + (ty)/(tx) = e^{y/x} + y/x = M(x, y)$ , and  $N(tx, ty) = -1 = N(x, y)$ . Therefore, the differential equation is homogeneous.

Let  $y = ux$ . Substituting into the differential equation,

$$u dx + x du = (e^u + u) dx, \quad \text{or} \quad e^{-u} du = \frac{dx}{x}.$$

Integrating and substituting for  $u$ , the final answer is

$$y(x) = -x \ln(C - \ln|x|).$$

## Section 1.4

1. Since  $M(x, y) = y^2 - x^2$ , and  $N(x, y) = 2xy$ ,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y^2 - x^2,$$

then  $u(x, y) = xy^2 - \frac{1}{3}x^3 + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = 2xy + f'(y) = 2xy.$$

Therefore,  $f'(y) = 0$ , and  $u(x, y) = xy^2 - \frac{1}{3}x^3 = C$ .

2. Since  $M(x, y) = y - x$ , and  $N(x, y) = x + y$ ,

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y - x,$$

then  $u(x, y) = xy - \frac{1}{2}x^2 + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = x + f'(y) = x + y.$$

Therefore,  $f'(y) = y$ , and  $u(x, y) = xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 = C$ .

3. Since  $M(x, y) = y^2 - 1$ , and  $N(x, y) = 2xy - \sin(y)$ ,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y^2 - 1,$$

then  $u(x, y) = xy^2 - x + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = 2xy + f'(y) = 2xy - \sin(y).$$

Therefore,  $f'(y) = -\sin(y)$ , and  $u(x, y) = xy^2 - x + \cos(y) = C$ .

4. Since  $M(x, y) = \sin(y) - 2xy + x^2$ , and  $N(x, y) = x \cos(y) - x^2$ ,

$$\frac{\partial M}{\partial y} = \cos(y) - 2x = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \sin(y) - 2xy + x^2,$$

then  $u(x, y) = x \sin(y) - x^2y + \frac{1}{3}x^3 + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = x \cos(y) - x^2 + f'(y) = x \cos(y) - x^2.$$

Therefore,  $f'(y) = 0$ , and  $u(x, y) = x \sin(y) - x^2y + \frac{1}{3}x^3 = C$ .

5. Since  $M(x, y) = -y/x^2$ , and  $N(x, y) = 1/x + 1/y$ ,

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x}.$$



The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2},$$

then  $u(x, y) = y/x + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = \frac{1}{x} + f'(y) = \frac{1}{x} + \frac{1}{y}.$$

Therefore,  $f'(y) = 1/y$ , and  $u(x, y) = y/x + \ln(y) = C$ .

6. Since  $M(x, y) = 3x^2 - 6xy$ , and  $N(x, y) = -3x^2 - 2y$ ,

$$\frac{\partial M}{\partial y} = -6x = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = 3x^2 - 6xy,$$

then  $u(x, y) = x^3 - 3x^2y + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = -3x^2 + f'(y) = -3x^2 - 2y.$$

Therefore,  $f'(y) = -2y$ , and  $u(x, y) = x^3 - 3x^2y - y^2 = C$ .

7. Since  $M(x, y) = y \sin(xy)$ , and  $N(x, y) = x \sin(xy)$ ,

$$\frac{\partial M}{\partial y} = \sin(xy) + xy \cos(xy) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y \sin(xy),$$

then  $u(x, y) = -\cos(xy) + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = x \sin(xy) + f'(y) = x \sin(xy).$$

Therefore,  $f'(y) = 0$ , and  $u(x, y) = -\cos(xy) = C$ .

8. Since  $M(x, y) = 2xy^2 + 3x^2$ , and  $N(x, y) = 2x^2y$ ,

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = 2xy^2 + 3x^2,$$

then  $u(x, y) = x^2y^2 + x^3 + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = 2x^2y + f'(y) = 2x^2y.$$

Therefore,  $f'(y) = 0$ , and  $u(x, y) = x^2y^2 + x^3 = C$ .

9. Since  $M(x, y) = 2xy^3 + 5x^4y$ , and  $N(x, y) = 3x^2y^2 + x^5 + 1$ ,

$$\frac{\partial M}{\partial y} = 6xy^2 + 5x^4 = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = 2xy^3 + 5x^4y,$$

then  $u(x, y) = x^2y^3 + x^5y + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = 3x^2y^2 + x^5 + f'(y) = 3x^2y^2 + x^5 + 1.$$

Therefore,  $f'(y) = 1$ , and  $u(x, y) = x^2y^3 + x^5y + y = C$ .

10. Since  $M(x, y) = x^3 + y/x$ , and  $N(x, y) = y^2 + \ln(x)$ ,

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = x^3 + \frac{y}{x},$$

then  $u(x, y) = \frac{1}{4}x^4 + y \ln(x) + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = \ln(x) + f'(y) = y^2 + \ln(x).$$

Therefore,  $f'(y) = y^2$ , and  $u(x, y) = \frac{1}{4}x^4 + y \ln(x) + \frac{1}{3}y^3 = C$ .

11.  $[x + e^{-y} + x \ln(y)] dy + [y \ln(y) + e^x] dx = 0$

Since  $M(x, y) = y \ln(y) + e^x$ , and  $N(x, y) = x + e^{-y} + x \ln(y)$ ,

$$\frac{\partial M}{\partial y} = 1 + \ln(y) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = y \ln(y) + e^x,$$

then  $u(x, y) = xy \ln(y) + e^x + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = x[\ln(y) + 1] + f'(y) = x + e^{-y} + x \ln(y).$$

Therefore,  $f'(y) = e^{-y}$ , and  $u(x, y) = xy \ln(y) + e^x - e^{-y} = C$ .

12. Since  $M(x, y) = \cos(4y^2)$ , and  $N(x, y) = -8xy \sin(4y^2)$ ,

$$\frac{\partial M}{\partial y} = -8y \sin(4y^2) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \cos(4y^2),$$

then  $u(x, y) = x \cos(4y^2) + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = -8xy \sin(4y^2) + f'(y) = -8xy \sin(4y^2).$$

Therefore,  $f'(y) = 0$ , and  $u(x, y) = x \cos(4y^2) = C$ .

13. Since  $M(x, y) = \sin^2(x + y)$ , and  $N(x, y) = -\cos^2(x + y)$ ,

$$\frac{\partial M}{\partial y} = 2 \sin(x + y) \cos(x + y) = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \sin^2(x + y) = \frac{1}{2} [1 - \cos(2x + 2y)],$$

then  $u(x, y) = x/2 - \sin(2x + 2y)/4 + f(y)$ . To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = -\frac{1}{2} \cos(2x + 2y) + f'(y) = -\cos^2(x + y) = -\frac{1}{2} [1 + \cos(2x + 2y)].$$

Therefore,  $f'(y) = -\frac{1}{2}$ , and  $u(x, y) = y - x + \frac{1}{2} \sin(2x + 2y) = C$ .

14. After multiplying by the integrating factor,

$$M(x, y) = \alpha \frac{y^{a+1}}{(1-y)^{a+1}}, \quad \text{and} \quad N(x, y) = (x-y) \frac{y^a}{(1-y)^{a+2}}.$$

Checking the exactness criteria,

$$\frac{\partial M}{\partial y} = \frac{y^a}{(1-y)^{a+2}} = \frac{\partial N}{\partial x}.$$

The exactness criteria is satisfied.

Now, since

$$\frac{\partial u}{\partial x} = \alpha \frac{y^{a+1}}{(1-y)^{a+1}},$$

$$u(x, y) = \alpha x \frac{y^{a+1}}{(1-y)^{a+1}} + f(y) = C.$$

To find  $f(y)$ , we use

$$\frac{\partial u}{\partial y} = x \frac{y^a}{(1-y)^{a+2}} + f'(y) = (x-y) \frac{y^a}{(1-y)^{a+2}}.$$

Therefore,

$$f'(y) = -\frac{y^{a+1}}{(1-y)^{a+2}}.$$

Integrating, we find that

$$f(y) = -\int_0^y \frac{\xi^{a+1}}{(1-\xi)^{a+2}} d\xi.$$

and the final answer is

$$u(x, y) = \alpha x \frac{y^{a+1}}{(1-y)^{a+1}} - \int_0^y \frac{\xi^{a+1}}{(1-\xi)^{a+2}} d\xi = C.$$

## Section 1.5

1. Since  $P(x) = 1$ ,  $\mu(x) = e^x$ . Multiplying the differential equation by the integrating, we have that  $e^x y' + e^x y = e^{2x}$ , or  $d(e^x y)/dx = e^{2x}$  or  $e^x y = \frac{1}{2} e^{2x} + C$ , or  $y = \frac{1}{2} e^x + C e^{-x}$ . This general solution applies to any  $x$  on the interval  $(-\infty, \infty)$ .

2. Since  $P(x) = 2x$ ,  $\mu(x) = e^{x^2}$ . Multiplying the differential equation by the integrating factor, we have that  $e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$ ,  $d(e^{x^2}y)/dx = xe^{x^2}$ , or  $e^{x^2}y = \frac{1}{2}e^{x^2} + C$ , or  $y = \frac{1}{2} + Ce^{-x^2}$ . This general solution applies to any  $x$  on the interval  $(-\infty, \infty)$ .

3. Since the canonical form of the differential equation is  $y' + y/x = 1/x^2$ ,  $P(x) = 1/x$ , and  $\mu(x) = x$ . Multiplying the canonical differential equation by the integrating factor, we have that  $xy' + y = x^{-1}$ , or  $d(xy)/dx = x^{-1}$ , or  $xy = \ln(x) + C$ , or  $y = \ln(x)/x + Cx^{-1}$ . This general solution applies to any  $x$  as long as  $x \neq 0$ .

4. Since the canonical form of the differential equation is  $y' - 2y/x = x$ ,  $P(x) = -2/x$ , and  $\mu(x) = x^{-2}$ . Multiplying the canonical differential equation by the integrating factor, we have that  $x^{-2}y' - 2x^{-3}y = x^{-1}$ ,  $d(y/x)/dx = 1/x$ , or  $y/x^2 = \ln(x) + C$ , or  $y = 2x \ln(x) + Cx^2$ . This general solution applies to any  $x$  on the interval  $(-\infty, \infty)$ .

5. Directly from the differential equation, we have that  $P(x) = -3/x$ , and  $\mu(x) = x^{-3}$ . Multiplying the differential equation by the integrating factor, we have that  $x^{-3}y' - 3x^{-4}y = 2x^{-1}$ ,  $d(y/x^3)/dx = 2/x$ , or  $y/x^3 = 2 \ln(x) + C$ , or  $y = 2x^3 \ln(x) + Cx^3$ . This general solution applies to any  $x$  on the interval  $(-\infty, \infty)$ .

6. Since  $P(x) = 2$ ,  $\mu(x) = e^{2x}$ . Multiplying the differential equation by the integrating factor, we have that  $e^{2x}y' + 2e^{2x}y = 2e^{2x} \sin(x)$ , or  $d(e^{2x}y)/dx = 2e^{2x} \sin(x)$ , or  $e^{2x}y = \frac{2}{5}e^{2x} [2 \sin(x) - \cos(x)] + C$ , or  $y = \frac{4}{5} \sin(x) - \frac{2}{5} \cos(x) + Ce^{-2x}$ . This general solution applies to any  $x$  on the interval  $(-\infty, \infty)$ .

7. Since the differential equation is already in canonical form, we immediately have  $P(x) = 2 \cos(2x)$ , and  $\mu(x) = \exp[\sin(2x)]$ . Multiplying the differential equation by the integrating factor, we have that  $e^{\sin(2x)}dy/dx + 2 \cos(2x)e^{\sin(2x)}y = 0$ ,  $d[e^{\sin(2x)}y]/dx = 0$ , or  $e^{\sin(2x)}y = C$ . This general solution applies to any  $x$  on the interval  $n\pi + \varphi < 2x < (n+1)\pi + \varphi$ , where  $\varphi$  is any real and  $n$  is any integer.

8. Dividing through by  $x$ , we immediately have  $P(x) = 1/x$ , and  $\mu(x) = x$ . Multiplying the differential equation by the integrating factor, we have that  $xdy/dx + y = \ln(x)$ , or  $d(xy)/dx = \ln(x)$ , or  $xy = C + x \ln(x) - x$ , or  $y = C/x + \ln(x) - 1$ . This general solution applies to any  $x$  on the interval  $(0, \infty)$ .

9. Since the differential equation is already in canonical form, we immediately have  $P(x) = 3$ , and  $\mu(x) = e^{3x}$ . Multiplying the differential equation by the integrating factor, we have that  $e^{3x}dy/dx + 3e^{3x}y = 4e^{3x}$ , or  $d(e^{3x}y)/dx =$

$4e^{3x}$ , or  $e^{3x}y(x) - y(0) = \frac{4}{3}(e^{3x} - 1)$ , or  $e^{3x}y(x) - 5 = \frac{4}{3}(e^{3x} - 1)$ , or  $y(x) = \frac{4}{3} + \frac{11}{3}e^{-3x}$ . This particular solution applies to any  $x$  on the interval  $(-\infty, \infty)$ .

10. Since the differential equation is already in canonical form, we immediately have  $P(x) = -1$ , and  $\mu(x) = e^{-x}$ . Multiplying the differential equation by the integrating factor, we have that  $e^{-x}dy/dx - e^{-x}y = 1/x$ , or  $d(e^{-x}y)/dx = 1/x$ , or  $e^{-x}y(x) - e^{-e}y(e) = \ln(x) - \ln(e)$ , or  $y(x) = e^x [\ln(x) - 1]$ .

11. By inspection, we immediately have that  $d[\sin(x)y]/dx = 1$ , or  $\sin(x)y = x + C$ , or  $y(x) = (x + C)/\sin(x)$ .

12. Since the canonical form of the differential equation is

$$\frac{dy}{dx} + \frac{2 \sin(x)}{1 - \cos(x)}y = \frac{\tan(x)}{1 - \cos(x)},$$

$P(x) = 2 \sin(x)/[1 - \cos(x)]$ , and  $\mu(x) = [1 - \cos(x)]^2$ . Multiplying the canonical differential equation by the integrating factor, we have that

$$\begin{aligned} [1 - \cos(x)]^2 \frac{dy}{dx} + 2 \sin(x)[1 - \cos(x)]y &= \tan(x)[1 - \cos(x)] \\ \frac{d}{dx} \{ [1 - \cos(x)]^2 y \} &= \tan(x) - \sin(x) \\ [1 - \cos(x)]^2 y &= -\ln |\cos(x)| + \cos(x) + C. \end{aligned}$$

This general solution applies to any  $x$  on the interval  $n\pi + \varphi < x < (n+1)\pi + \varphi$ , where  $\varphi$  is any real and  $n$  is any integer.

13. Since the differential equation is already in canonical form,  $P(x) = a \tan(x) + b \sec(x)$ , and

$$\mu(x) = \frac{[\sec(x) + \tan(x)]^b}{\cos^a(x)}.$$

Multiplying by the integrating factor, we have that

$$\frac{d}{dx} \left\{ \frac{[\sec(x) + \tan(x)]^b}{\cos^a(x)} y(x) \right\} = c \frac{\sec(x)[\sec(x) + \tan(x)]^b}{\cos^a(x)}.$$

Integrating both sides of this equation,

$$\frac{[\sec(x) + \tan(x)]^b}{\cos^a(x)} y(x) - y(0) = c \int_0^x \frac{[\sec(\xi) + \tan(\xi)]^b}{\cos^{a+1}(\xi)} d\xi$$

or

$$y(x) = \frac{\cos^a(x) y(0)}{[\sec(x) + \tan(x)]^b} + \frac{c \cos^a(x)}{[\sec(x) + \tan(x)]^b} \int_0^x \frac{[\sec(\xi) + \tan(\xi)]^b}{\cos^{a+1}(\xi)} d\xi.$$

14. Writing the differential equation in canonical form, we have

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}.$$

Therefore,

$$\mu(x) = \exp\left[\int^x \left(1 + \frac{1}{\xi}\right) d\xi\right] = \exp[x + \ln(x)] = xe^x.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned} xe^x \frac{dy}{dx} + (x+1)e^x y &= e^x \\ \frac{d}{dx} [xe^x y] &= e^x \\ xe^x y &= e^x + C \\ y &= \frac{1}{x} + Ce^{-x}. \end{aligned}$$

15. Since the differential equation is already in canonical form,  $P(x) = 2a$ , and  $\mu(x) = e^{2ax}$ . Multiplying by the integrating factor, we have that

$$\frac{d}{dx} [e^{2ax} y(x)] = \frac{x}{2} e^{2ax} - \frac{\sin(2\omega x)}{4\omega} e^{2ax}.$$

Integrating both sides of this equation,

$$e^{2ax} y(x) - y(0) = \frac{1}{2} \int_0^x \xi e^{2a\xi} d\xi - \frac{1}{4\omega} \int_0^x \sin(2\omega\xi) e^{2a\xi} d\xi$$

or

$$e^{2ax} y(x) = \frac{e^{2a\xi}}{8a^2} (2a\xi - 1) \Big|_0^x - \frac{e^{2a\xi}}{8\omega(a^2 + \omega^2)} [a \sin(2\omega\xi) - \omega \cos(2\omega\xi)] \Big|_0^x.$$

Solving for  $y(x)$ ,

$$y(x) = \frac{2ax - 1}{8a^2} + \frac{\omega^2 e^{-2ax}}{8a^2(a^2 + \omega^2)} - \frac{a \sin(2\omega x) - \omega \cos(2\omega x)}{8\omega(a^2 + \omega^2)}.$$

16. Since the differential equation is already in canonical form,  $P(x) = 2k/x^3$ , and  $\mu(x) = \exp(-k/x^2)$ . Multiplying by the integrating factor, we have that

$$\frac{d}{dx} [e^{-k/x^2} y(x)] = \ln\left(\frac{x+1}{x}\right) e^{-k/x^2}.$$

Integrating both sides of this equation,

$$e^{-k/x^2} y(x) - e^{-k} y(1) = \int_1^x \ln\left(\frac{\xi+1}{\xi}\right) \exp\left(-\frac{k}{\xi^2}\right) d\xi.$$

Because  $y(1) = 0$ ,

$$y(x) = \exp\left(\frac{k}{x^2}\right) \int_1^x \ln\left(\frac{\xi+1}{\xi}\right) \exp\left(-\frac{k}{\xi^2}\right) d\xi.$$

17. Substituting in the variable  $p(x) = y^2(x)$ , we have that

$$\frac{dp}{dx} - \frac{2}{kx} p = -\frac{2}{k}, \quad p(1) = 0.$$

Then, multiplying through with the integrating factor, we find that

$$\begin{aligned} x^{-2/k} \frac{dp}{dx} - \frac{2}{k} x^{-2/k-1} p &= -\frac{2}{k} x^{-2/k} \\ \frac{d}{dx} [x^{-2/k} p] &= -\frac{2}{k} x^{-2/k}. \end{aligned}$$

If  $k \neq 2$ , an integration yields

$$x^{-2/k} p(x) = -\frac{2/k}{1-2/k} x^{1-2/k} + C, \quad \text{or} \quad y^2(x) = -\frac{2}{k-2} x + C x^{2/k}.$$

Applying the initial condition, the final answer is

$$y^2(x) = \frac{2}{2-k} \left(x - x^{2/k}\right),$$

provided  $k \neq 2$ . If  $k = 2$ , then we have that

$$\frac{p(x)}{x} = -\ln(x) + C.$$

Applying the initial condition,  $C = 0$  and the final answer is

$$y^2(x) = x \ln(1/x).$$

18. We must solve

$$\frac{dx}{dt} - (1-N)rx = S, \quad x(0) = 1.$$



Multiplying both side of the equation by  $e^{-(1-N)rt}$ , we have

$$\frac{d}{dt} \left[ e^{-(1-N)rt} x(t) \right] = S e^{-(1-N)rt}.$$

Integrating both sides of this equation from 0 to  $t$ , we obtain

$$e^{-(1-N)rt} x(t) - 1 = \frac{S}{(1-N)r} \left[ 1 - e^{-(1-N)rt} \right].$$

Solving  $x(t)$ , we find that

$$x(t) = e^{(1-N)rt} + \frac{S}{(1-N)r} \left[ e^{(1-N)rt} - 1 \right].$$

19. From separation of variables,

$$\frac{d[A]}{[A]} = -k_1 dt.$$

Integration yields

$$[A] = [A]_0 e^{-k_1 t}.$$

Substituting  $[A]$  into the equation for  $[B]$ ,

$$\frac{d[B]}{dt} + k_2[B] = k_1[A]_0 e^{-k_1 t}.$$

Multiplying by the integrating factor, we have that

$$\frac{d}{dt} (e^{k_2 t} [B]) = k_1 [A]_0 e^{(k_2 - k_1)t}.$$

Integrating this equation, we find that

$$e^{k_2 t} [B] - [B]_0 = \frac{k_1 [A]_0}{k_2 - k_1} \left[ e^{(k_2 - k_1)t} - 1 \right].$$

Because  $[B]_0 = 0$ , we find that

$$[B] = \frac{k_1 [A]_0}{k_2 - k_1} \left[ e^{-k_1 t} - e^{-k_2 t} \right].$$

Finally, substituting for  $[B]$  into the  $[C]$  equation,

$$\frac{d[C]}{dt} = \frac{k_1 k_2 [A]_0}{k_2 - k_1} \left[ e^{-k_1 t} - e^{-k_2 t} \right].$$

Integrating this equation,

$$[C] - [C]_0 = \frac{k_2[A]_0}{k_2 - k_1} (1 - e^{-k_1 t}) - \frac{k_1[A]_0}{k_2 - k_1} (1 - e^{-k_2 t})$$

or

$$[C] = [A]_0 \left( 1 + \frac{k_1 e^{-k_2 t} - k_2 e^{-k_1 t}}{k_2 - k_1} \right).$$

since  $[C]_0 = 0$ .

20. The differential equation is

$$L \frac{dI}{dt} + RI = E_0 \cos^2(\omega t), \quad I(0) = 0.$$

Multiplying the integrating factor, we can rewrite this differential equation

$$\frac{d}{dt} [e^{Rt/L} I(t)] = \frac{E_0}{L} e^{Rt/L} \cos^2(\omega t).$$

Integrating both sides of the differential equation,

$$e^{Rt/L} I(t) - I(0) = \frac{E_0}{L} \int_0^t e^{R\tau/L} \cos^2(\omega \tau) d\tau.$$

Using the fact that  $I(0) = 0$  and replacing  $\cos^2(\omega t)$  with  $\frac{1}{2} [1 + \cos(2\omega t)]$ , we have that

$$e^{Rt/L} I(t) = \frac{E_0}{2L} \int_0^t [e^{R\tau/L} + e^{R\tau/L} \cos(2\omega \tau)] d\tau,$$

or

$$e^{Rt/L} I(t) = \frac{E_0 (e^{R\tau/L} - 1)}{2R} + \frac{E_0 \{e^{R\tau/L} [R \cos(2\omega \tau) + 2\omega L \sin(2\omega \tau)] - R\}}{2R^2 + 8\omega^2 L^2}.$$

Solving for  $I(t)$ ,

$$I(t) = \frac{E_0 (1 - e^{-Rt/L})}{2R} + \frac{E_0 \{R [\cos(2\omega \tau) - e^{-Rt/L}] + 2\omega L \sin(2\omega \tau)\}}{2R^2 + 8\omega^2 L^2}.$$

21. Because  $n = 2$ ,  $z = y^{-1}$  and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{z}{x} = 1, \quad \text{or} \quad \frac{1}{x} \frac{dz}{dx} - \frac{z}{x^2} = \frac{d}{dx} \left( \frac{z}{x} \right) = \frac{1}{x}.$$

Integrating this equation yields

$$z(x) = Cx + x \ln(x).$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \frac{1}{Cx + x \ln(x)}.$$

22. Because  $n = 2$ ,  $z = y^{-1}$  and the linear ordinary differential equation is

$$\frac{dz}{dx} + \frac{z}{x} = -\frac{1}{x^2}, \quad \text{or} \quad x \frac{dz}{dx} + z = \frac{d}{dx}(xz) = -\frac{1}{x}.$$

Integrating this equation yields

$$z(x) = [C - \ln(x)]/x.$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \frac{x}{C - \ln(x)}.$$

23. Because  $n = \frac{1}{2}$ ,  $z = y^{1/2}$  and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{2z}{x} = \frac{x}{2}, \quad \text{or} \quad \frac{1}{x^2} \frac{dz}{dx} - \frac{2z}{x^3} = \frac{d}{dx} \left( \frac{z}{x^2} \right) = \frac{1}{2x}.$$

Integrating this equation yields

$$z(x) = Cx^2 + \frac{1}{2}x^2 \ln(x).$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = [Cx^2 + \frac{1}{2}x^2 \ln(x)]^2.$$

24. Because  $n = 2$ ,  $z = y^{-1}$  and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{z}{x} = x, \quad \text{or} \quad \frac{1}{x} \frac{dz}{dx} - \frac{z}{x^2} = \frac{d}{dx} \left( \frac{z}{x} \right) = 1.$$

Integrating this equation yields

$$z(x) = Cx + x^2.$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = \frac{1}{Cx + x^2}.$$

25. Because  $n = -1$ ,  $z = y^2$  and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{z}{x} = -1, \quad \text{or} \quad \frac{1}{x} \frac{dz}{dx} - \frac{z}{x^2} = \frac{d}{dx} \left( \frac{z}{x} \right) = -\frac{1}{x}.$$

Integrating this equation yields

$$z(x) = Cx - x \ln(x).$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = [Cx - x \ln(x)]^{1/2}.$$

26. Because  $n = 3$ ,  $z = y^{-2}$  and the linear ordinary differential equation is

$$\frac{dz}{dx} - \frac{2z}{x} = -1$$

or

$$\frac{1}{x^2} \frac{dz}{dx} - \frac{2z}{x^3} = \frac{d}{dx} \left( \frac{z}{x^2} \right) = -\frac{1}{x^2}.$$

Integrating this equation yields

$$z(x) = Cx^2 + x.$$

Therefore, the solution to the nonlinear differential equation is

$$y(x) = [Cx^2 + x]^{-1/2}.$$

## Section 1.6

5. The equilibrium points for this differential equation are  $x = 0, \frac{1}{2}$ , and 1. The right side is negative for  $0 < x < \frac{1}{2}$  and positive for  $\frac{1}{2} < x < 1$ . Thus, the equilibrium point  $x = \frac{1}{2}$  is unstable while the equilibrium points at  $x = 0$  and 1 are stable.

6. The equilibrium points are  $x = \pm 1$ , and  $x = \pm 2$ . For  $x < -2$ ,  $-1 < x < 1$ , and  $x > 2$ ,  $x' > 0$ . For  $-2 < x < -1$  and  $1 < x < 2$ ,  $x' < 0$ . Therefore, the equilibrium points at  $x = -2$  and  $x = 1$  are stable while  $x = -1$  and  $x = 2$  are unstable.

7. There is only one equilibrium point,  $x = 0$ . For  $x < 0$ ,  $x' > 0$  while for  $x > 0$ ,  $x' < 0$ . Therefore, this equilibrium point is stable.
8. The equilibrium points are  $x = \pm 2$ , and  $x = 0$ . For  $x < -2$  and  $0 < x < 2$ ,  $x' > 0$ . For  $-2 < x < 0$  and  $x > 2$ ,  $x' < 0$ . Therefore, the equilibrium point  $x = 0$  is unstable while the points  $x = \pm 2$  are stable.

Section 1.7

1. Because the differential equation can be written  $x' - x = t$ , we have that

$$\frac{d}{dt} (e^{-t}x) = -te^{-t}.$$

Integration yields  $x(t) = Ce^t + t + 1$ . Applying the initial condition, we find that  $C = 1$ . Therefore, the final answer is  $x(t) = e^t + t + 1$ .

2. Because the differential equation can be written  $dx/x = t dt$ ,  $x(t) = Ce^{t^2/2}$ . Applying the initial condition,  $C = 1$ . Therefore, the final solution is  $x(t) = e^{t^2/2}$ .

3. Because the differential equation can be written  $dx/x^2 = dt/(t + 1)$ , integration yields  $1/x = C - \ln(t + 1)$ . Applying the initial condition,  $C = 1$ . Therefore, the final answer is  $x(t) = [1 - \ln(t + 1)]^{-1}$ .

4. Because the differential equation can be written  $x' - x = e^{-t}$ , we have that

$$\frac{d}{dt} (e^{-t}x) = e^{-2t}.$$

Integration yields  $x(t) = Ce^t - \frac{1}{2}e^{-t}$ . Applying the initial condition, we find that  $C = e^{-2}/2$ . Therefore, the final answer is  $x(t) = \frac{1}{2} (e^{t-2} - e^{-t})$ .

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%           MATLAB Code for Differential-Integral Equation
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initialize parameters
clear; beta = 0.4; deltat = 0.01; K = 1000;

% vary value of b

for n = 1:4

    b = 0.2 * (n-1);
```

```
% take the first time step

x(1) = 0; t(1) = 0; t(2) = deltat;
x(2) = x(1) + deltat - b * deltat * sign(x(1)) * abs(x(1))^beta;
sum = x(1) + x(2);

% take the remaining time steps

for k = 2:K
    t(k) = t(k-1) + deltat;
    x(k) = x(k-1) + deltat ...
        - b * deltat * sign(x(k-1)) * abs(x(k-1))^beta ...
        - deltat * deltat * sum;
    sum = sum + x(k);
end

% plot the results

subplot(2,2,n), plot(t,x); xlabel('time','FontSize',20);
ylabel('x(t)','FontSize',20); legend(['B = ',num2str(b)])

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

**Section 2.0**

1. Since the second solution is  $y_2(x) = u(x)$ ,  $y_2'(x) = u'(x)$  and  $y_2''(x) = u''(x)$ . Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $xu'' + 2u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = -\frac{2}{x}, \quad u'(x) = Cx^{-2}.$$

Thus,  $u(x) = A/x$ , and the second solution is  $y_2(x) = A/x$ .

2. Since the second solution is  $y_2(x) = u(x)e^x$ ,

$$y_2'(x) = u'(x)e^x + u(x)e^x, \quad \text{and} \quad y_2''(x) = u''(x)e^x + 2u'(x)e^x + u(x)e^x.$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $u'' + 3u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = -3, \quad u'(x) = Ce^{-3x}.$$

Thus,  $u(x) = Ae^{-3x}$ , and the second solution is  $y_2(x) = Ae^{-2x}$ .

3. Since the second solution is  $y_2(x) = xu(x)$ ,

$$y_2'(x) = xu'(x) + u(x), \quad \text{and} \quad y_2''(x) = xu''(x) + 2u'(x).$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $xu'' + 6u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = -\frac{6}{x}, \quad u'(x) = Cx^{-6}.$$

Thus,  $u(x) = Ax^{-5}$ , and the second solution is  $y_2(x) = Ax^{-4}$ .

4. Since the second solution is  $y_2(x) = e^x u(x)$ ,

$$y_2'(x) = e^x u(x) + e^x u'(x), \quad \text{and} \quad y_2''(x) = e^x u(x) + 2e^x u'(x) + e^x u''(x).$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $xu'' + (x-1)u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{1}{x} - 1, \quad u'(x) = -Axe^{-x}.$$

Thus,  $u(x) = A(x+1)e^{-x}$ , and the second solution is  $y_2(x) = A(x+1)$ .

5. Since the second solution is  $y_2(x) = (x-1)u(x)$ ,

$$y_2'(x) = (x-1)u'(x) + u(x), \quad \text{and} \quad y_2''(x) = (x-1)u''(x) + 2u'(x).$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $x(2-x)(x-1)u'' + 2u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{2}{x(x-2)(x-1)} = \frac{1}{x} + \frac{1}{x-2} - \frac{2}{x-1},$$

and

$$u'(x) = A \frac{x(x-2)}{(x-1)^2} = A \left[ 1 - \frac{1}{(x-1)^2} \right].$$

Thus,  $u(x) = Ax + A/(x-1)$ , and the second solution is  $y_2(x) = A(x^2 - x + 1)$ .

6. Since the second solution is  $y_2(x) = u(x) \sin^3(x)$ ,

$$y_2'(x) = 3 \sin^2(x) \cos(x) u(x) + \sin^3(x) u'(x),$$

and

$$y_2''(x) = 6 \sin(x) \cos^2(x) u(x) - 3 \sin^3(x) u(x) \\ + 6 \sin^2(x) \cos(x) u'(x) + \sin^3(x) u''(x).$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that

$$\sin(x) \cos(x) u'' + [6 \cos^2(x) + \sin^2(x)] u' = 0.$$

Therefore,

$$\frac{u''(x)}{u'(x)} = -6 \cot(x) - \tan(x), \quad u'(x) = -\frac{A \cos(x)}{5 \sin^6(x)}.$$

Thus,  $u(x) = A/\sin^5(x)$ , and the second solution is  $y_2(x) = A/\sin^2(x)$ .

7. Since the second solution is  $y_2(x) = u(x) \cos(x)/\sqrt{x}$ ,

$$y_2'(x) = -\frac{\sin(x)}{\sqrt{x}} u(x) - \frac{\cos(x)}{2x\sqrt{x}} u(x) + \frac{\cos(x)}{\sqrt{x}} u'(x)$$

and

$$y_2''(x) = -\frac{\cos(x)}{\sqrt{x}} u(x) + \frac{3 \cos(x)}{4x^2\sqrt{x}} u(x) + \frac{\sin(x)}{x\sqrt{x}} u(x) \\ - \frac{\cos(x)}{x\sqrt{x}} u'(x) - \frac{2 \sin(x)}{\sqrt{x}} u'(x) + \frac{\cos(x)}{\sqrt{x}} u''(x).$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $\cos(x)u'' - 2 \sin(x)u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = \frac{2 \sin(x)}{\cos(x)}, \quad u'(x) = A \sec^2(x).$$

Thus,  $u(x) = A \tan(x)$ , and the second solution is  $y_2(x) = A \sin(x)/\sqrt{x}$ .

8. Since the second solution is  $y_2(x) = u(x)e^{-bx^2/2}$

$$y_2'(x) = -bx e^{-bx^2/2} u(x) + e^{-bx^2/2} u'(x),$$

and

$$y_2''(x) = (b^2 x^2 - b) e^{-bx^2/2} u(x) - 2bx e^{-bx^2/2} u'(x) + u''(x) e^{-bx^2/2}.$$

Consequently, substituting these values of  $y_2(x)$ ,  $y_2'(x)$ , and  $y_2''(x)$  into the differential equation, we find that  $u'' + (a - 2bx)u' = 0$ . Therefore,

$$\frac{u''(x)}{u'(x)} = 2bx - a.$$



Thus,  $u(x) = \int^x e^{b\xi^2 - a\xi} d\xi$ , and the second solution is

$$y_2(x) = e^{-bx^2/2} \int^x e^{b\xi^2 - a\xi} d\xi.$$

9. Letting  $v = y'$ , we can rewrite the differential equation  $y(dv/dy) = v^2$ . Assuming  $v \neq 0$ , we can integrate this equation to give  $v = C_1 y$ . Therefore,  $dy/dx = C_1 y$ . Integrating this, we obtain the final answer  $y(x) = C_2 e^{C_1 x}$ .

10. Letting  $v = y'$ , we can rewrite the differential equation  $dv/dy = 2y$ . Assuming  $v \neq 0$ , we can integrate this equation to give  $v = y^2 + C_1$ . Because  $v(1) = 1$ ,  $C_1 = 0$  and  $dy/dx = y^2$ . Integrating this equation,  $1/y = C_2 - x$ . Again, using the initial conditions,  $y(x) = 1/(1 - x)$ .

11. Letting  $v = y'$ , we can rewrite the differential equation  $yv(dv/dy) = v + v^2$ . Assuming  $v \neq 0$ , we can integrate this equation to give  $1 + v = C_1 y$ . Substituting for  $v$ ,  $dy/dx = C_1 y - 1$ . Integrating this equation,  $y = (1 + C_2 e^{C_1 x}) / C_1$ .

12. Letting  $v = y'$ , we can rewrite the differential equation  $2yv(dv/dy) = 1 + v^2$ . Assuming  $v \neq 0$ , we can integrate this equation to give  $1 + v^2 = C_1 y$ . Substituting for  $v$ ,  $dy/dx = \sqrt{C_1 y - 1}$ . Integrating this equation,  $y = [1 + (C_1 x + C_2)^2 / 4] / C_1$ .

13. Letting  $v = y'$ , we can rewrite the differential equation  $v(dv/dy) = e^{2y}$  with  $v(0) = 1$ . We can integrate this equation and find  $v = e^y$ . Substituting for  $v$ ,  $dy/dx = e^y$ . Integrating this equation,  $e^{-y} = C - x$ , or  $y = -\ln|1 - x|$ .

14. We begin by integrating once and using the initial conditions,  $y'' = \frac{3}{2}y^2$ . Letting  $v = y'$ , we can rewrite this differential equation  $v(dv/dy) = \frac{3}{2}y^2$ . Integrating this equation, we have  $v = y^{3/2}$ . Substituting for  $v$ ,  $dy/dx = y^{3/2}$ . Integrating this equation,  $y = 4/(2 - x)^2 = 4/(x - 2)^2$ .

15. If we define  $z = 1/v$ , the Bernoulli equation becomes  $z' + z/x = -\frac{1}{2}$ . Its solution is  $z = -A^2/x - x/4$ . Therefore,  $y' = v = -4x/(x^2 + 4A^2)$ . Integration yields the final answer  $y(x) = B - 2\ln(x^2 + 4A^2)$ .

16. First we compute

$$y'(x) = u'(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] - \frac{a_1}{2a_2} u(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right],$$

and

$$\begin{aligned} y''(x) = & u''(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] - \frac{1}{2} \frac{d}{dx} \left(\frac{a_1}{a_2}\right) u(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] \\ & - \frac{a_1}{a_2} u'(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right] + \frac{a_1^2}{4a_2^2} u(x) \exp\left[-\frac{1}{2} \int^x \frac{a_1(\xi)}{a_2(\xi)} d\xi\right]. \end{aligned}$$

Substituting  $y(x)$ ,  $y'(x)$ , and  $y''(x)$  into the original ordinary differential equation yields the final answer.

### Section 2.1

1. The characteristic equation is  $m^2 + 6m + 5 = (m+1)(m+5) = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{-x} + C_2 e^{-5x}$ .
2. The characteristic equation is  $m^2 - 6m + 10 = (m-3+i)(m-3-i) = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{3x} \cos(x) + C_2 e^{3x} \sin(x)$ .
3. The characteristic equation is  $m^2 - 2m + 1 = (m-1)^2 = 0$ . Therefore, the general solution is  $y(x) = C_1 e^x + C_2 x e^x$ .
4. The characteristic equation is  $m^2 - 3m + 2 = (m-1)(m-2) = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{2x} + C_2 e^x$ .
5. The characteristic equation is  $m^2 - 4m + 8 = (m-2)^2 + 4 = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{2x} \cos(2x) + C_2 e^{2x} \sin(2x)$ .
6. The characteristic equation is  $m^2 + 6m + 9 = (m+3)^2 = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{-3x} + C_2 x e^{-3x}$ .
7. The characteristic equation is  $m^2 + 6m - 40 = (m+10)(m-4) = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{-10x} + C_2 e^{4x}$ .
8. The characteristic equation is  $m^2 + 4m + 5 = (m+2)^2 + 1 = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{-2x} \cos(x) + C_2 e^{-2x} \sin(x)$ .
9. The characteristic equation is  $m^2 + 8m + 25 = (m+4)^2 + 9 = 0$ . Therefore, the general solution is  $y(x) = e^{-4x} [C_1 \cos(3x) + C_2 \sin(3x)]$ .
10. The characteristic equation is  $4m^2 - 12m + 9 = (2m-3)^2 = 0$ . Therefore, the general solution is  $y(x) = e^{3x/2} (C_1 + C_2 x)$ .
11. The characteristic equation is  $m^2 + 8m + 16 = (m+4)^2 = 0$ . Therefore, the general solution is  $y(x) = C_1 e^{-4x} + C_2 x e^{-4x}$ .
12. The characteristic equation is  $m^3 + 4m^2 = m^2(m+4) = 0$ . Therefore, the general solution is  $y(x) = C_1 + C_2 x + C_3 e^{-4x}$ .
13. The characteristic equation is  $m^4 + 4m^2 = m^2(m^2 + 4) = 0$ . Therefore, the general solution is  $y(x) = C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$ .
14. The characteristic equation is  $m^4 + 2m^3 + m^2 = m^2(m+1)^2 = 0$ . Therefore, the general solution is  $y(x) = C_1 + C_2 x + C_3 e^{-x} + C_4 x e^{-x}$ .