PART III

SAMPLE PROBLEM SOLUTIONS

CHAPTER ONE

Modeling Change

1.1 SOLUTIONS

1. a. $\{1, 3, 9, 27, 81\}$
b. $\{0, 6, 18, 42, 90\}$

3. a. $a_{n+1} = a_n + 2, a_0 = 2$
b. $a_{n+1} = a_n^2, a_0 = 2$

5.
a.

$$\begin{aligned} a_1 &= 3a_0 = 3 \\ a_2 &= 3a_1 = 9 \\ a_3 &= 3a_2 = 27 \\ a_4 &= 3a_3 = 81 \end{aligned}$$

b.

$$\begin{aligned} a_1 &= 2a_0 + 6 = 6 \\ a_2 &= 2a_1 + 6 = 18 \\ a_3 &= 2a_2 + 6 = 42 \\ a_4 &= 2a_3 + 6 = 90 \end{aligned}$$

7. Let a_n be the amount in the account after n months.

$$\begin{aligned} a_{n+1} &= a_n + 0.005a_n + 200 \\ a_0 &= 5000 \end{aligned}$$

8. Let a_n be the amount owed after n months.

$$\begin{aligned} a_{n+1} &= a_n + 0.015a_n - 50 \\ a_0 &= 500 \end{aligned}$$

9. Let a_n be the amount owed after n months.

$$\begin{aligned} a_{n+1} &= a_n + 0.005a_n - p \\ a_0 &= 200,000 \\ a_{360} &= 0 \end{aligned}$$

10. Let a_n be the value of the account after n months.

$$a_{n+1} = a_n + 0.01a_n - 1000$$
$$a_0 = 50000$$

For depleting the account, find n such that $a_n = 0.$ The account will be depleted after time period 69, $a_{69} = 655.28.$

11. Let a_n be the value of the account after n months.$a_{n+1} = 0.005a_n - 1000, a_0 = 50000.$ The account will be depleted after time period 57, $a_{57} = 677.29.$

1.2 SOLUTIONS

3. Let a_n be the number of people who have the information after n days.

$$a_{n+1} = a_n + a_n(N - a_n).$$

Here we assume that the increase in the number of people with the information is the product of the number with the information and those without the information.

4. Let a_n be the number of people infected after n intervals. Then $(N - a_n)$ are not infected. If we assume the increase in the number infected is the product of those infected and those not infected, we have the following model:

$$a_{n+1} = a_n + a_n(N - a_n).$$

6. Let a_n be the concentration of drug after n hours. Then,

$$a_{n+1} - a_n = -0.2a_n$$
$$a_{n+1} = 0.8a_n$$
$$a_0 = 640$$

The concentration reaches 100 between the 8th and 9th hour.

8. Let a_n be the percent of Carbon-14 remaining after n intervals of 5700 years.

$$a_{n+1} = 0.5a_n$$
$$a_0 = 1$$

Find n such that $a_n = 0.01.$The value of $n = 6.6$ intervals or $(6.6) \cdot 5700 = 37,620$ years

1.3 SOLUTIONS

1. a. 3^n

b. $10 \cdot 5^n$

c. $64 \cdot \left(\frac{3}{4}\right)^n$

d. $2 \cdot (2)^n + 1$

e. $-2 \cdot (-1)^n + 1$

f. $\frac{-203}{90} \cdot \left(\frac{1}{10}\right)^n + \frac{32}{9}$

2. a. $a = 0$, unstable

b. $a = 0$, stable

d. constant solutions-every value is an equilibrium value.

e. $a = 22\frac{8}{11}$, unstable

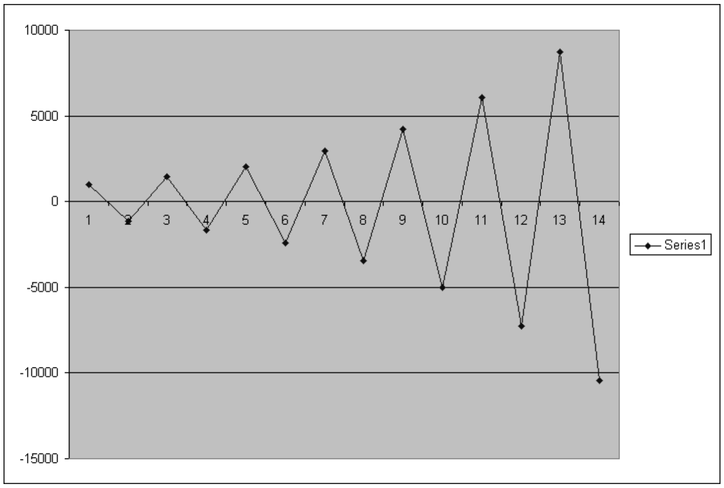
g. $a = \frac{100}{0.2} = 500$, stable

j. no equilibrium value

3. a. The equilibrium value $a = 22\frac{8}{11}$ is unstable

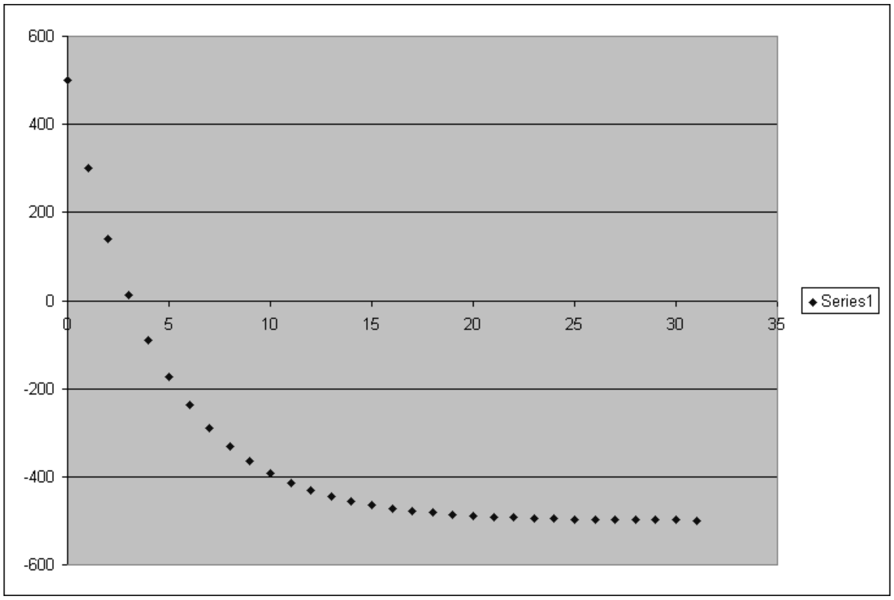
n	a(n)	a(n)	a(n)
0	1000	22.727	0
1	-1150	22.727	50
2	1430	22.727	-10
3	-1666	22.727	62
4	2049.2	22.727	-24.4
5	-2409	22.727	79.28
6	2940.8	22.727	-45.14
7	-3479	22.727	104.16
8	4224.8	22.727	-75
9	-5020	22.727	140
10	6073.7	22.727	-118
11	-7238	22.727	191.59
12	8736.2	22.727	-179.9
13	-10433	22.727	265.89

EQUIL 22.72727Not stable



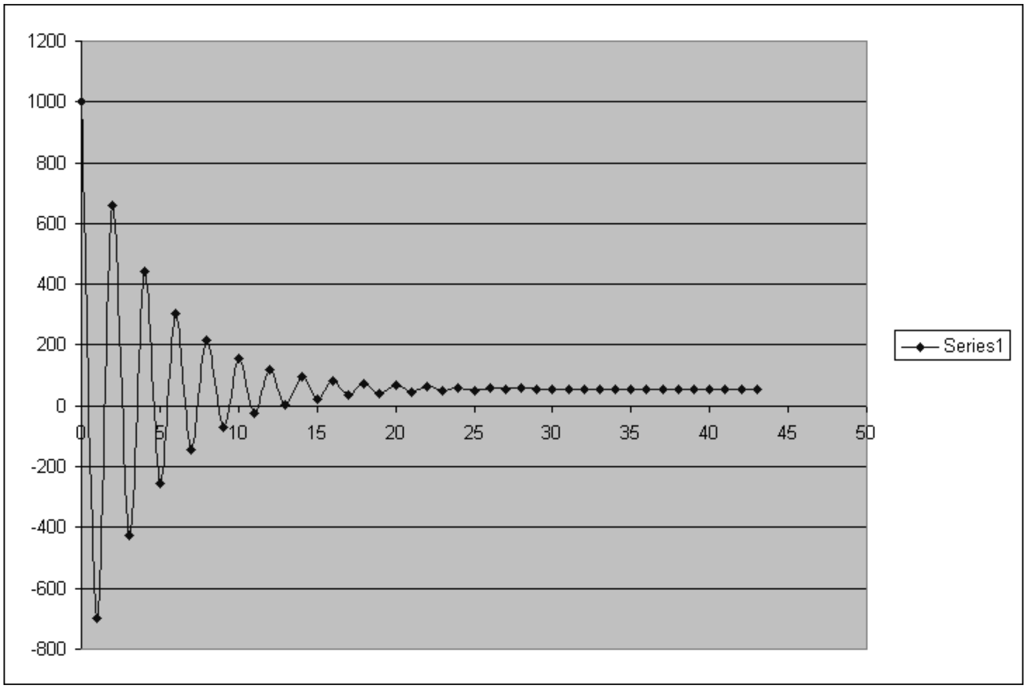
b. The equilibrium value $a = -500$ is stable.

n	a(n)
0	500
1	300
2	140
3	12
4	-90.4
5	-172.32
6	-237.856
7	-290.285
8	-332.228
9	-365.782
10	-392.626
11	-414.101
12	-431.281
13	-445.024
14	-456.02
15	-464.816
16	-471.853
17	-477.482
18	-481.986
19	-485.588
20	-488.471
21	-490.777
22	-492.621
23	-494.097
24	-495.278
25	-496.222
26	-496.978
27	-497.582
28	-498.066
29	-498.453



c. The equilibrium value $a = -500$ is stable.

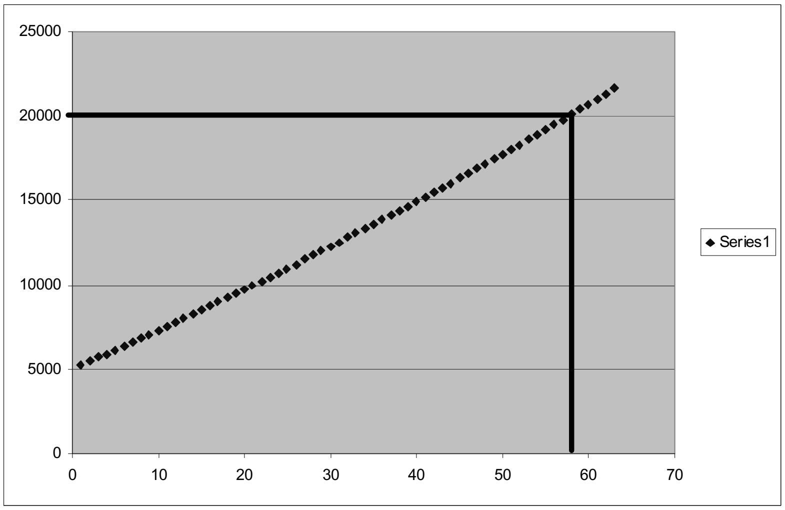
d. Equilibrium value is $a = 55.5555$ or $100/(1 + .8)$. It is stable.



5.

$$a_{n+1} = a_n + 0.005a_n + 200$$
$$a_0 = 5000$$

After 58 months, the value of the account is $20095.80.



50	17745.16167
51	18033.88748
52	18324.05692
53	18615.6772
54	18908.75559
55	19203.29937
56	19499.31586
57	19796.81244
58	20095.7965
59	20396.27549
60	20698.25686

6.

$$a_{n+1} = a_n + 0.015a_n - 50$$
$$a_0 = 500$$

The equilibrium value $a = \frac{-50}{-.015} = 3333.33$ is unstable. At the equilibrium value, the payment equals the interest. The debt is paid in 11 months with the final payment= $1.015(45.13) = 45.81$.

n	a(n)
0	500
1	457.5
2	414.3625

3	370.5779375
4	326.1366066
5	281.0286557
6	235.2440855
7	188.7727468
8	141.604338
9	93.72840305
10	45.1343291
11	-4.188655968

7.

$$a_{n+1} = a_n + 0.005a_n - p$$
$$a_0 = 100,000$$
$$a_{360} = 0$$

If $p \approx \$599.55,$ the loan will be paid in 360 months.

9.

$$a_{n+1} = a_n + 0.01a_n - 1000$$
$$a_0 = 50000$$

At the equilibrium value $a = \frac{-1000}{-0.01} = 100,000$ the withdrawal of \$1000 equals the interest earned. The equilibrium value is unstable. The account is depleted after 70 months.

10.

$$a_{n+1} = 0.69a_n + 0.1$$
$$a_0 = 0.5$$

The equilibrium value $a = \frac{0.1}{0.31} = 0.32258$ which is stable represents the "steady-state" or long-term amount of digoxin in the bloodstream.

13. All 1,000 people have heard the rumor after period 11.

1.4 SOLUTIONS

3. **a.** $B_{n+1} = B_n - .05 \cdot F_n$
$F_{n+1} = F_n - 0.15 \cdot B_n$
$B_0 = 27, F_0 = 33$

b. After 10 stages, the French-Spanish have less than 1 ships remaining, and the British have 18.43 ships. The French-Spanish have lost about 33 ships while the British about 8 ships.

Period	British Ships	French-Spanish Ships
0	27	33
1	25.35	28.95
2	23.905	25.15
3	22.65	21.56
4	21.57	18.17
5	20.66	14.93
6	19.91	11.83
7	19.32	8.84
8	18.88	5.95
9	18.58	3.11
10	18.43	0.327

c.

Period	British Ships	French-Spanish Ships
Battle A		
0	13	3
1	12.85	1.05
Battle B		
0	26.85	18.05
1	25.95	14.02
2	25.25	10.13
3	24.74	6.34
4	24.42	2.63
Battle C		
0	24.42	15.63
1	23.64	11.97
2	23.04	8.42
3	22.62	4.97
4	22.37	1.57

Battle A concludes after approximately Stage 1. The British have 12.85 ships to combine with the 14 in reserve giving the British 26.85 to begin the battle with Force B. We will assume that the 1.05 ships remaining in Force A join Force B.

In battle B, we will assume the battle ends after 4 stages with Force B having 2.63 ships, and the British 24.42 ships. Again we assume that the remaining ships from Battle B join Battle C.

We estimate that the battle is over after 4 stages with the French-Spanish having 1.57 ships left ships remaining, and the British 22.37 ships remaining.

Using Lord Nelson's strategy, the British have lost about 4 ships instead of 27, and the French-Spanish have lost about 31 instead of 15. The new strategy and new technology was effective.

9. The equilibrium is $R = 0, I = 0$, and $S = 0$.

CHAPTER TWO

The Modeling Process, Proportionality, and Geometric Similarity

2.1 SOLUTIONS

1. Population growth of a single species is presented in Section 11.1 and you may wish to refer to that section of the text for a more thorough discussion than is given here.

Let's restrict our attention to a single species and identify the problem as follows: *For a given species with a known current population, predict the population P at some future time.*

For a single species, population growth depends on the birth rate, the death rate, and the availability of resources. The birth rate depends on the kind of species, the habitat, food supply, crowding conditions (some species produce fewer off-spring when overcrowding occurs), predators (some species with many predators will produce more off-spring), the general health of the species population, the number of females, and so forth. (If the species is a human population, the birth rate is influenced by such factors as infant mortality rate, attitudes toward and availability of contraceptives, attitudes toward abortion, health care during pregnancy, and so forth.) The death rate depends on the species, the number of predators in the habitat, the health of the species, availability of food and water, environmental factors, and so forth. (For human populations, the death rate is affected by sanitation and public health, war, pollution, diet, medicines, psychological stress and anxiety, age and gender, distribution of the current population, and so forth. Other factors affecting human population growth in a particular region include immigration and emigration, living space restrictions, and epidemics.) We can see that many factors influence population growth.

Initially, we might assume that the rate of change of the population over time is proportional to the present population level. This assumption leads to the differential equation $\frac{dP}{dt} = kP$, where k is a proportionality factor. When $k > 0$, the population is growing; for $k < 0$, the population is on the decline.

A simple model might assume that k is a constant. It is probably more realistic to assume that k varies with the population P. As the population increases, for example, competition for a limited food supply and living space will cause k to decrease, so that k is a decreasing function of P. It may also be true that k is time dependent. The assumptions that $k > 0$ is a constant and that $k = r(M - P)$ for $r > 0$ and $M > 0$ constant, are investigated in Section 11.1.

2. Let's assume that the retail store is a discount department store such as K-Mart which is located within the inner city. We identify the following problem: *minimize the cost of procuring, installing, and operating a lighting system while meeting acceptable standards***.** (Alternatively, the store may wish to maximize the lighting possible at a given cost level , including procurement, installation, operation, and maintenance. The store may wish to take into account traffic patterns in the parking lot so that those areas exhibiting the greatest traffic flow and parking utilization are best illuminated.) We will assume that the size and shape of the parking lot has already been determined and is fixed. A question to answer is during what hours the store desires to operate the lighting system. If the management desires to retain some lighting after closing hours to deter crime, how much lighting is desired and where in the parking lot?