1.1 Introduction to Systems of Linear Equations

1. **(a)** This is a linear equation in x_1, x_2, and x_3.

(b) This is not a linear equation in x_1, x_2, and x_3 because of the term x_1x_3.

(c) We can rewrite this equation in the form $x_1 + 7x_2 - 3x_3 = 0$ therefore it is a linear equation in x_1, x_2, and x_3.

(d) This is not a linear equation in x_1, x_2, and x_3 because of the term x_1^{-2}.

(e) This is not a linear equation in x_1, x_2, and x_3 because of the term $x_1^{3/5}$.

(f) This is a linear equation in x_1, x_2, and x_3.

2. **(a)** This is a linear equation in x and y.

(b) This is not a linear equation in x and y because of the terms $2x^{1/3}$ and $3\sqrt{y}$.

(c) This is a linear equation in x and y.

(d) This is not a linear equation in x and y because of the term $\frac{\pi}{7}\cos x$.

(e) This is not a linear equation in x and y because of the term xy.

(f) We can rewrite this equation in the form $-x + y = -7$ thus it is a linear equation in x and y.

3. **(a)**
$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

(b)
$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$

(c)
$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & = & b_2 \end{array}$$

4. **(a)**
$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

(c)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \end{bmatrix}$$

5. **(a)**

$$\begin{aligned} 2x_1 \quad\quad\quad &= 0 \\ 3x_1 - 4x_2 &= 0 \\ x_2 &= 1 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 \quad\quad - 2x_3 &= 5 \\ 7x_1 + x_2 + 4x_3 &= -3 \\ -2x_2 + x_3 &= 7 \end{aligned}$$

6. **(a)**

$$\begin{aligned} 3x_2 - x_3 - x_4 &= -1 \\ 5x_1 + 2x_2 \quad\quad - 3x_4 &= -6 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 \quad\quad + x_3 - 4x_4 &= 3 \\ -4x_1 \quad\quad + 4x_3 + x_4 &= -3 \\ -x_1 + 3x_2 \quad\quad - 2x_4 &= -9 \\ -x_4 &= -2 \end{aligned}$$

7. **(a)** $\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$

8. **(a)** $\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

9. The values in (a), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system. The 3-tuples in (b) and (c) are not solutions of the system.

10. The values in (b), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system. The 3-tuples in (a) and (c) are not solutions of the system.

11. **(a)** We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{aligned} 3x - 2y &= 4 \\ 0 &= 1 \end{aligned}$$

The second equation is contradictory, so the original system has no solutions. The lines represented by the equations in that system have no points of intersection (the lines are parallel and distinct).

(b) We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{aligned} 2x - 4y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restriction on x and y therefore we can omit it. The lines represented by the original system have infinitely many points of intersection. Solving the first equation for x we obtain $x=\frac{1}{2}+2y$. This allows us to represent the solution using parametric equations

$$x=\frac{1}{2}+2t, \qquad y=t$$

where the parameter t is an arbitrary real number.

(c) We can eliminate x from the second equation by adding -1 times the first equation to the second. This yields the system

$$\begin{aligned} x - 2y &= 0 \\ -2y &= 8 \end{aligned}$$

From the second equation we obtain $y=-4$. Substituting -4 for y into the first equation results in $x=-8$. Therefore, the original system has the unique solution

$$x=-8, \qquad y=-4$$

The represented by the equations in that system have one point of intersection: $(-8,-4)$.

12. We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{aligned} 2x - 3y &= a \\ 0 &= b-2a \end{aligned}$$

If $b-2a=0$ (i.e., $b=2a$) then the second equation imposes no restriction on x and y; consequently, the system has infinitely many solutions.

If $b-2a\neq 0$ (i.e., $b\neq 2a$) then the second equation becomes contradictory thus the system has no solutions.

There are no values of a and b for which the system has one solution.

13. (a) Solving the equation for x we obtain $x=\frac{3}{7}+\frac{5}{7}y$ therefore the solution set of the original equation can be described by the parametric equations

$$x=\frac{3}{7}+\frac{5}{7}t, \qquad y=t$$

where the parameter t is an arbitrary real number.

(b) Solving the equation for x_1 we obtain $x_1=\frac{7}{3}+\frac{5}{3}x_2-\frac{4}{3}x_3$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1=\frac{7}{3}+\frac{5}{3}r-\frac{4}{3}s, \qquad x_2=r, \qquad x_3=s$$

where the parameters r and s are arbitrary real numbers.

(c) Solving the equation for x_1 we obtain $x_1 = -\frac{1}{8} + \frac{1}{4}x_2 - \frac{5}{8}x_3 + \frac{3}{4}x_4$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = -\frac{1}{8} + \frac{1}{4}r - \frac{5}{8}s + \frac{3}{4}t, \quad x_2 = r, \quad x_3 = s, \quad x_4 = t$$

where the parameters r, s, and t are arbitrary real numbers.

(d) Solving the equation for v we obtain $v = \frac{8}{3}w - \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3}z$ therefore the solution set of the original equation can be described by the parametric equations

$$v = \frac{8}{3}t_1 - \frac{2}{3}t_2 + \frac{1}{3}t_3 - \frac{4}{3}t_4, \quad w = t_1, \quad x = t_2, \quad y = t_3, \quad z = t_4$$

where the parameters t_1, t_2, t_3, and t_4 are arbitrary real numbers.

14. **(a)** Solving the equation for x we obtain $x = 2 - 10y$ therefore the solution set of the original equation can be described by the parametric equations

$$x = 2 - 10t, \quad y = t$$

where the parameter t is an arbitrary real number.

(b) Solving the equation for x_1 we obtain $x_1 = 3 - 3x_2 + 12x_3$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = 3 - 3r + 12s, \quad x_2 = r, \quad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

(c) Solving the equation for x_1 we obtain $x_1 = 5 - \frac{1}{2}x_2 - \frac{3}{4}x_3 - \frac{1}{4}x_4$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = 5 - \frac{1}{2}r - \frac{3}{4}s - \frac{1}{4}t, \quad x_2 = r, \quad y = s, \quad z = t$$

where the parameters r, s, and t are arbitrary real numbers.

(d) Solving the equation for v we obtain $v = -w - x + 5y - 7z$ therefore the solution set of the original equation can be described by the parametric equations

$$v = -t_1 - t_2 + 5t_3 - 7t_4, \quad w = t_1, \quad x = t_2, \quad y = t_3, \quad z = t_4$$

where the parameters t_1, t_2, t_3, and t_4 are arbitrary real numbers.

15. **(a)** We can eliminate x from the second equation by adding -3 times the first equation to the second. This yields the system

$$\begin{aligned} 2x - 3y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restriction on x and y therefore we can omit it. Solving the first equation for x we obtain $x = \frac{1}{2} + \frac{3}{2}y$. This allows us to represent the solution using parametric equations

$$x = \frac{1}{2} + \frac{3}{2}t, \qquad y = t$$

where the parameter t is an arbitrary real number.

(b) We can see that the second and the third equation are multiples of the first: adding -3 times the first equation to the second, then adding the first equation to the third yields the system

$$\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for x_1 we obtain $x_1 = -4 - 3x_2 + x_3$. This allows us to represent the solution using parametric equations

$$x_1 = -4 - 3r + s, \quad x_2 = r, \quad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

16. **(a)** We can eliminate x_1 from the first equation by adding -2 times the second equation to the first. This yields the system

$$\begin{aligned} 0 &= 0 \\ 3x_1 + x_2 &= -4 \end{aligned}$$

The first equation does not impose any restriction on x_1 and x_2 therefore we can omit it. Solving the second equation for x_1 we obtain $x_1 = -\frac{4}{3} - \frac{1}{3}x_2$. This allows us to represent the solution using parametric equations

$$x_1 = -\frac{4}{3} - \frac{1}{3}t, \quad x_2 = t$$

where the parameter t is an arbitrary real number.

(b) We can see that the second and the third equation are multiples of the first: adding -3 times the first equation to the second, then adding 2 times the first equation to the third yields the system

$$\begin{aligned} 2x - y + 2z &= -4 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for x we obtain $x = -2 + \frac{1}{2}y - z$. This allows us to represent the solution using parametric equations

$$x=-2+\frac{1}{2}r-s,\quad y=r,\quad z=s$$

where the parameters r and s are arbitrary real numbers.

17. **(a)** Add 2 times the second row to the first to obtain $\begin{bmatrix}1&-7&8&8\\2&-3&3&2\\0&2&-3&1\end{bmatrix}$.

(b) Add the third row to the first to obtain $\begin{bmatrix}1&3&-8&3\\2&-9&3&2\\1&4&-3&3\end{bmatrix}$

(another solution: interchange the first row and the third row to obtain $\begin{bmatrix}1&4&-3&3\\2&-9&3&2\\0&-1&-5&0\end{bmatrix}$).

18. **(a)** Multiply the first row by $\frac{1}{2}$ to obtain $\begin{bmatrix}1&2&-3&4\\7&1&4&3\\-5&4&2&7\end{bmatrix}$.

(b) Add the third row to the first to obtain $\begin{bmatrix}1&-1&-3&6\\3&-1&8&1\\-6&3&-1&4\end{bmatrix}$

(another solution: add -2 times the second row to the first to obtain $\begin{bmatrix}1&-2&-18&0\\3&-1&8&1\\-6&3&-1&4\end{bmatrix}$).

19. **(a)** Add -4 times the first row to the second to obtain $\begin{bmatrix}1&k&-4\\0&8-4k&18\end{bmatrix}$ which corresponds to the system

$$x+\qquad ky=-4$$

$$(8-4k)y=18$$

If $k=2$ then the second equation becomes $0=18$, which is contradictory thus the system becomes inconsistent.

If $k\neq 2$ then we can solve the second equation for y and proceed to substitute this value into the first equation and solve for x.

Consequently, for all values of $k\neq 2$ the given augmented matrix corresponds to a consistent linear system.

(b) Add -4 times the first row to the second to obtain $\begin{bmatrix}1&k&-1\\0&8-4k&0\end{bmatrix}$ which corresponds to the system

$$\begin{aligned} x + \quad ky &= -1 \\ (8-4k)y &= 0 \end{aligned}$$

If $k = 2$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq 2$ then the second equation yields $y = 0$ and the first equation becomes $x = -1$.

Consequently, for all values of k the given augmented matrix corresponds to a consistent linear system.

20. **(a)** Add 2 times the first row to the second to obtain $\begin{bmatrix} 3 & -4 & k \\ 0 & 0 & 2k+5 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} 3x - 4y &= k \\ 0 &= 2k+5 \end{aligned}$$

If $k = -\frac{5}{2}$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq -\frac{5}{2}$ then the second equation is contradictory thus the system becomes inconsistent.

Consequently, the given augmented matrix corresponds to a consistent linear system only when $k = -\frac{5}{2}$.

(b) Add the first row to the second to obtain $\begin{bmatrix} k & 1 & -2 \\ 4+k & 0 & 0 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} kx + y &= -2 \\ (4+k)x &= 0 \end{aligned}$$

If $k = -4$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq -4$ then the second equation yields $x = 0$ and the first equation becomes $y = -2$.

Consequently, for all values of k the given augmented matrix corresponds to a consistent linear system.

21. Substituting the coordinates of the first point into the equation of the curve we obtain

$$y_1 = ax_1^2 + bx_1 + c$$

Repeating this for the other two points and rearranging the three equations yields

$$\begin{aligned} x_1^2 a + x_1 b + c &= y_1 \\ x_2^2 a + x_2 b + c &= y_2 \\ x_3^2 a + x_3 b + c &= y_3 \end{aligned}$$

This is a linear system in the unknowns a, b, and c. Its augmented matrix is $\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$.

23. Solving the first equation for x_1 we obtain $x_1 = c - kx_2$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = c - kt, \qquad x_2 = t$$

where the parameter t is an arbitrary real number.

Substituting these into the second equation yields

$$c - kt + lt = d$$

which can be rewritten as

$$c - kt = d - lt$$

This equation must hold true for all real values t, which requires that the coefficients associated with the same power of t on both sides must be equal. Consequently, $c = d$ and $k = l$.

24. **(a)** The system has no solutions if either

- at least two of the three lines are parallel and distinct or
- each pair of lines intersects at a different point (without any lines being parallel)

(b) The system has exactly one solution if either

- two lines coincide and the third one intersects them or
- all three lines intersect at a single point (without any lines being parallel)

(c) The system has infinitely many solutions if all three lines coincide.

25.

$$\begin{aligned} 2x + 3y + z &= 7 \\ 2x + y + 3z &= 9 \\ 4x + 2y + 5z &= 16 \end{aligned}$$

26. We set up the linear system as discussed in Exercise 21:

$$\begin{aligned} 1^2 a + 1b + c &= 1 \\ 2^2 a + 2b + c &= 4 \\ (-1)^2 a - 1b + c &= 1 \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} a + b + c &= 1 \\ 4a + 2b + c &= 4 \\ a - b + c &= 1 \end{aligned}$$

One solution is expected, since exactly one parabola passes through any three given points (x_1, y_1), (x_2, y_2), (x_3, y_3) if x_1, x_2, and x_3 are distinct.

27.

$$\begin{aligned} x + y + z &= 12 \\ 2x + y + 2z &= 5 \\ -x + z &= 1 \end{aligned}$$

True-False Exercises

(a) True. $(0,0,\ldots,0)$ is a solution.

(b) False. Only multiplication by a **non**zero constant is a valid elementary row operation.

(c) True. If $k=6$ then the system has infinitely many solutions; otherwise the system is inconsistent.

(d) True. According to the definition, $a_1x_1+a_2x_2+\cdots+a_nx_n=b$ is a linear equation if the *a*'s are not all zero. Let us assume $a_j \neq 0$. The values of all *x*'s except for x_j can be set to be arbitrary parameters, and the equation can be used to express x_j in terms of those parameters.

(e) False. E.g. if the equations are all homogeneous then the system must be consistent. (See True-False Exercise (a) above.)

(f) False. If $c \neq 0$ then the new system has the same solution set as the original one.

(g) True. Adding -1 times one row to another amounts to the same thing as subtracting one row from another.

(h) False. The second row corresponds to the equation $0=-1$, which is contradictory.

1.2 Gaussian Elimination

1. **(a)** This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(b) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(c) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(d) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(e) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(f) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(g) This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (-7) above it. The matrix is in row echelon form but not reduced row echelon form.

2. **(a)** This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (2) above it. The matrix is in row echelon form but not reduced row echelon form.

(b) This matrix does not have property 1 since its first nonzero number in the third row (2) is not a 1. The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(c) This matrix has properties 1-3 but does not have property 4: the third column contains a leading 1 and a nonzero number (4) above it. The matrix is in row echelon form but not reduced row echelon form.

(d) This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (5) above it. The matrix is in row echelon form but not reduced row echelon form.

(e) This matrix does not have property 2 since the row that consists entirely of zeros is not at the bottom of the matrix. The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(f) This matrix does not have property 3 since the leading 1 in the second row is directly below the leading 1 in the first (instead of being farther to the right). The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(g) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

3. **(a)** The first three columns are pivot columns and all three rows are pivot rows. The linear system

$$\begin{array}{rcrcrcl} x & - & 3y & + & 4z & = & 7 \\ & & y & + & 2z & = & 2 \\ & & & & z & = & 5 \end{array} \quad \text{can be rewritten as} \quad \begin{array}{rcl} x & = & 7+3y-4z \\ y & = & 2-2z \\ z & = & 5 \end{array}$$

and solved by back-substitution:

$$\begin{aligned} &z=5 \\ &y=2-2(5)=-8 \\ &x=7+3(-8)-4(5)=-37 \end{aligned}$$

therefore the original linear system has a unique solution: $x=-37$, $y=-8$, $z=5$.

(b) The first three columns are pivot columns and all three rows are pivot rows. The linear system

$$\begin{array}{rcrcrcrcl} w & & & + & 8y & - & 5z & = & 6 \\ & & x & + & 4y & - & 9z & = & 3 \\ & & & & y & + & z & = & 2 \end{array} \quad \text{can be rewritten as} \quad \begin{array}{rcl} w & = & 6-8y+5z \\ x & = & 3-4y+9z \\ y & = & 2-z \end{array}$$

Let $z=t$. Then

$$\begin{aligned} &y=2-t \\ &x=3-4(2-t)+9t=-5+13t \\ &w=6-8(2-t)+5t=-10+13t \end{aligned}$$

therefore the original linear system has infinitely many solutions:

$$w=-10+13t,\ x=-5+13t,\ y=2-t,\ z=t$$

where t is an arbitrary value.

(c) Columns 1, 3, and 4 are pivot columns. The first three rows are pivot rows. The linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 7x_2 & - & 2x_3 & & & - & 8x_5 & = & -3 \\ & & & & x_3 & + & x_4 & + & 6x_5 & = & 5 \\ & & & & & & x_4 & + & 3x_5 & = & 9 \\ & & & & & & & & 0 & = & 0 \end{array}$$

can be rewritten: $x_1=-3-7x_2+2x_3+8x_5$, $x_3=5-x_4-6x_5$, $x_4=9-3x_5$.

Let $x_2=s$ and $x_5=t$. Then

$$x_4 = 9 - 3t$$
$$x_3 = 5 - (9 - 3t) - 6t = -4 - 3t$$
$$x_1 = -3 - 7s + 2(-4 - 3t) + 8t = -11 - 7s + 2t$$

therefore the original linear system has infinitely many solutions:

$$x_1 = -11 - 7s + 2t, \quad x_2 = s, \quad x_3 = -4 - 3t, \quad x_4 = 9 - 3t, \quad x_5 = t$$

where s and t are arbitrary values.

(d) The first two columns are pivot columns and the first two rows are pivot rows. The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

4. **(a)** The first three columns are pivot columns and all three rows are pivot rows. A unique solution: $x = -3$, $y = 0$, $z = 7$.

(b) The first three columns are pivot columns and all three rows are pivot rows. Infinitely many solutions: $w = 8 + 7t$, $x = 2 - 3t$, $y = -5 - t$, $z = t$ where t is an arbitrary value.

(c) Columns 1, 3, and 4 are pivot columns. The first three rows are pivot rows. Infinitely many solutions: $v = -2 + 6s - 3t$, $w = s$, $x = 7 - 4t$, $y = 8 - 5t$, $z = t$ where s and t are arbitrary values.

(d) Columns 1 and 3 are pivot columns. The first two rows are pivot rows. The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

5. $$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$ ← The first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$ ← -3 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$ ← The second row was multiplied by -1.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$ ← 10 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ ← The third row was multiplied by $-\frac{1}{52}$.

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ x_2 - 5x_3 &= -9 \\ x_3 &= 2 \end{aligned}$$

and can be rewritten as

$$\begin{aligned} x_1 &= 8 - x_2 - 2x_3 \\ x_2 &= -9 + 5x_3 \\ x_3 &= 2 \end{aligned}$$

Back-substitution yields

$$\begin{aligned} x_3 &= 2 \\ x_2 &= -9 + 5(2) = 1 \\ x_1 &= 8 - 1 - 2(2) = 3 \end{aligned}$$

The linear system has a unique solution: $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

6. $$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$$ ← 2 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$$ ← -8 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix}$$ ← The second row was multiplied by $\frac{1}{7}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ← 7 times the second row was added to the third row.

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + \frac{4}{7}x_3 &= \frac{1}{7} \\ 0 &= 0 \end{aligned}$$

Solve the equations for the leading variables

$$x_1 = -x_2 - x_3$$

$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

then substitute the second equation into the first

$$x_1 = -\frac{1}{7} - \frac{3}{7}x_3$$
$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$$x_1 = -\frac{1}{7} - \frac{3}{7}t, \quad x_2 = \frac{1}{7} - \frac{4}{7}t, \qquad x_3 = t$$

7. $\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$ ⟵ -2 times the first row was added to the second row.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$ ⟵ The first row was added to the third row.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$ ⟵ -3 times the first row was added to the fourth row.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$ ← The second row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$ ← -1 times the second row was added to the third row.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← -3 times the second row was added to the fourth row.

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x - y + 2z - w &= -1 \\ y - 2z &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Solve the equations for the leading variables

$$\begin{aligned} x &= -1 + y - 2z + w \\ y &= 2z \end{aligned}$$

then substitute the second equation into the first

$$\begin{aligned} x &= -1 + 2z - 2z + w = -1 + w \\ y &= 2z \end{aligned}$$

If we assign z and w the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x = -1 + t, \quad y = 2s, \qquad z = s, \qquad w = t$$

8. $$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The first and second rows were interchanged.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ← -6 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ← The second row was multiplied by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix}$$ ← 6 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{6}$.

The system of equations corresponding to this augmented matrix in row echelon form

$$\begin{aligned} a + 2b - c &= -\frac{2}{3} \\ b - \frac{3}{2}c &= -\frac{1}{2} \\ 0 &= 1 \end{aligned}$$

is clearly inconsistent.

9. $$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$ ← The first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$ ← -3 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$ ← The second row was multiplied by -1.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$ ← 10 times the second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← The third row was multiplied by $-\frac{1}{52}$.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← 5 times the third row was added to the second row.

$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -2 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -1 times the second row was added to the first row.

The linear system has a unique solution: $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

10. $\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$ ← The first row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix}$ ← 2 times the first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix}$ ← -8 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix}$ ← The second row was multiplied by $\frac{1}{7}$.

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← 7 times the second row was added to the third row.

$\begin{bmatrix} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← -1 times the second row was added to the first row.

Infinitely many solutions: $x_1 = -\frac{1}{7} - \frac{3}{7}t$, $x_2 = \frac{1}{7} - \frac{4}{7}t$, $x_3 = t$ where t is an arbitrary value.

11. $\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$ ← -2 times the first row was added to the second row.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$ ← the first row was added to the third row.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$ ← -3 times the first row was added to the fourth row.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$ ← The second row was multiplied by $\frac{1}{3}$.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$ ← -1 times the second row was added to the third row.

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← -3 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← the second row was added to the first row.

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x \qquad\qquad - w &= -1 \\ y - 2z \qquad &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Solve the equations for the leading variables

$$x = -1 + w$$

$$y = 2z$$

If we assign z and w the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x = -1 + t, \qquad y = 2s, \qquad z = s, \qquad w = t$$

12. $\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix}$ ← The augmented matrix for the system.

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The first and second rows were interchanged.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix}$$ ← The first row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ← -6 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix}$$ ← The second row was multiplied by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix}$$ ← 6 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ← The third row was multiplied by $\frac{1}{6}$.

$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ← $\frac{1}{2}$ times the third row was added to the second row.

$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ← $\frac{2}{3}$ times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ← -2 times the second row was added to the first row.

The last row corresponds to the equation

$$0a + 0b + 0c = 1$$

therefore the system is inconsistent.

(Note: this was already evident after the fifth elementary row operation.)

13. Since the number of unknowns (4) exceeds the number of equations (3), it follows from Theorem 1.2.2 that this system has infinitely many solutions. Those include the trivial solution and infinitely many nontrivial solutions.

14. The system does not have nontrivial solutions.
(The third equation requires $x_3 = 0$, which substituted into the second equation yields $x_2 = 0$. Both of these substituted into the first equation result in $x_1 = 0$.)

15. We present two different solutions.

Solution I uses Gauss-Jordan elimination

$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ ← The first row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ ← -1 times the first row was added to the second row.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ ← The second row was multiplied by $\frac{2}{3}$.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$ ← -1 times the second row was added to the third row.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← The third row was added to the second row and $-\frac{3}{2}$ times the third row was added to the first row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ← $-\frac{1}{2}$ times the second row was added to the first row.

Unique solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

Solution II. This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exact sequence of elementary row operations being used does not matter – see part 1 of the discussion "Some Facts About Echelon Forms" in Section 1.2)

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ ← The first and second rows were interchanged (to avoid introducing fractions into the first row).

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ ← -2 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ ← The second row was multiplied by $-\frac{1}{3}$.

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ ⟵ -1 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was added to the second row.

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -2 times the second row was added to the first row.

Unique solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

16. We present two different solutions.
Solution I uses Gauss-Jordan elimination

$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$ ⟵ The first row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$ ⟵ The first row was added to the second row.

$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix}$ ⟵ -1 times the first row was added to the third row.

$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{2}{3}$.

$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}$ ⟵ $-\frac{3}{2}$ times the second row was added to the third row.

$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was multiplied by $\frac{1}{10}$.

$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ 3 times the third row was added to the second row and $\frac{3}{2}$ times the third row was added to the first row

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ $\frac{1}{2}$ times the second row was added to the first row.

Unique solution: $x = 0$, $y = 0$, $z = 0$.

Solution II. This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exact sequence of elementary row operations being used does not matter – see part 1 of the discussion “Some Facts About Echelon Forms” in Section 1.2)

$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ -1 & 2 & -3 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$ ⟵ The first and third rows were interchanged (to avoid introducing fractions into the first row).

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$ ⟵ The first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & -11 & 0 \end{bmatrix}$ ⟵ -2 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{bmatrix}$ ⟵ The second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was multiplied by $-\frac{1}{10}$.

$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -1 times the third row was added to the second row.

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -4 times the third row was added to the first row.

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{1}{3}$.

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -1 times the second row was added to the first row.

Unique solution: $x = 0$, $y = 0$, $z = 0$.

17. $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ ⟵ The first row was multiplied by $\frac{1}{3}$.

$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{bmatrix}$ ⟵ -5 times the first row was added to the second row.

$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{3}{8}$.

$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$ ⟵ $-\frac{1}{3}$ times the second row was added to the first row.

If we assign x_3 and x_4 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x_1 = -\frac{1}{4}s, \quad x_2 = -\frac{1}{4}s - t, \quad x_3 = s, \quad x_4 = t.$$

(Note that fractions in the solution could be avoided if we assigned $x_3 = 4s$ instead, which along with $x_4 = t$ would yield $x_1 = -s$, $x_2 = -s - t$, $x_3 = 4s$, $x_4 = t$.)

18. $\begin{bmatrix} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ← The first and second rows were interchanged.

$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ← The first row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{bmatrix}$ ← -2 times the first row was added to the third row and 4 times the first row was added to the fourth row.

$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← -2 times the second row was added to the third row and the second row was added to the fourth row.

$\begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← $-\frac{1}{2}$ times the second row was added to the first row.

If we assign w and x the arbitrary values s and t, respectively, the general solution is given by the formulas

$$u = \frac{7}{2}s - \frac{5}{2}t, \quad v = -3s + 2t, \quad w = s, \quad x = t.$$

19. $\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$$ ← The first and second rows were interchanged.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix}$$ ← -2 times the first row was added to the third row and 2 times the first row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix}$$ ← The second row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix}$$ ← -3 times the second row was added to the third and -1 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← 10 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← -2 times the third row was added to the second and 3 times the third row was added to the first row.

If we assign y an arbitrary value t the general solution is given by the formulas

$$w = t, \quad x = -t, \quad y = t, \quad z = 0 .$$

20. $$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix}$$ ← -1 times the first row was added to the second row, -2 times the first row was added to the fourth row, and -1 times the first row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix}$$ ← 2 times the second row was added to the third row, 10 times the second row was added to the fourth row, and 5 times the second row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{41}{2} & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{bmatrix}$$ ← -21 times the third row was added to the fourth row and -9 times the third row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{bmatrix}$$ ← The fourth row was multiplied by $\frac{2}{41}$.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← $-\frac{17}{2}$ times the fourth row was added to the fifth row.

The augmented matrix in row echelon form corresponds to the system

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_2 + 2x_3 - x_4 &= 0 \\ x_3 - \frac{3}{2}x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

Using back-substitution, we obtain the unique solution of this system

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 .$$

21.

$$\begin{bmatrix} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 2 & -1 & 3 & 4 & 9 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \longleftarrow$$ The first and second rows were interchanged (to avoid introducing fractions into the first row).

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & -1 & 7 & -10 & -13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the second row, -3 times the first row was added to the third row, and -2 times the first row was added to the fourth.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix} \longleftarrow$$ The second row was multiplied by -1 .

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & -14 & 14 & 14 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix} \longleftarrow$$ 3 times the second row was added to the third row and -1 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix} \longleftarrow$$ The third row was multiplied by $-\frac{1}{14}$.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix} \longleftarrow$$ -15 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow$$ The fourth row was multiplied by $-\frac{1}{5}$.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & -7 & 0 & -7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ← The fourth row was added to the third row, -10 times the fourth row was added to the second, and -7 times the fourth row was added to the first.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ← 7 times the third row was added to the second row, and 2 times the third row was added to the first row.

Unique solution: $I_1 = -1$, $I_2 = 0$, $I_3 = 1$, $I_4 = 2$.

22.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$ ← The first and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix}$$ ← The first row was added to the second row and -2 times the first row was added to the last row.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix}$$ ← The second and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}$$ ← -3 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}$$ ← The third row was multiplied by $-\frac{1}{3}$.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← 3 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← -1 times the third row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← 2 times the second row was added to the first row.

If we assign Z_2 and Z_5 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$Z_1 = -s - t, \quad Z_2 = s, \quad Z_3 = -t, \quad Z_4 = 0, \quad Z_5 = t.$$

23. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has infinitely many solutions (the third unknown can be assigned an arbitrary value t, then back-substitution can be used to solve for the first two unknowns).

(c) The system is inconsistent since the third equation $0 = 1$ is contradictory.

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$ the system is consistent with infinitely many solutions.
- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$).

24. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has a unique solution (solve the first equation for the first unknown, then proceed to solve the second equation for the second unknown and solve the third equation last.)

(c) The system is inconsistent (adding -1 times the first row to the second yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$; the second equation $0 = 1$ is contradictory).

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ the system is consistent with infinitely many solutions.
- For $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

25.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$ ⟵ -3 times the first row was added to the second row and -4 times the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ ⟵ -1 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{1}{7}$.

The system has no solutions when $a=-4$ (since the third row of our last matrix would then correspond to a contradictory equation $0=-8$).

The system has infinitely many solutions when $a=4$ (since the third row of our last matrix would then correspond to the equation $0=0$).

For all remaining values of a (i.e., $a \neq -4$ and $a \neq 4$) the system has exactly one solution.

26.

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & -(a^2-3) & a \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & -a^2+2 & a-2 \end{bmatrix}$ ⟵ -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{2} \\ 0 & 0 & -a^2+2 & a-2 \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{1}{6}$.

The system has no solutions when $a=\sqrt{2}$ or $a=-\sqrt{2}$ (since the third row of our last matrix would then correspond to a contradictory equation).

For all remaining values of a (i.e., $a\neq\sqrt{2}$ and $a\neq-\sqrt{2}$) the system has exactly one solution.

There is no value of a for which this system has infinitely many solutions.

27. $\begin{bmatrix}1 & 3 & -1 & a\\ 1 & 1 & 2 & b\\ 0 & 2 & -3 & c\end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix}1 & 3 & -1 & a\\ 0 & -2 & 3 & -a+b\\ 0 & 2 & -3 & c\end{bmatrix}$ ← -1 times the first row was added to the second row.

$\begin{bmatrix}1 & 3 & -1 & a\\ 0 & -2 & 3 & -a+b\\ 0 & 0 & 0 & -a+b+c\end{bmatrix}$ ← The second row was added to the third row.

$\begin{bmatrix}1 & 3 & -1 & a\\ 0 & 1 & -\frac{3}{2} & \frac{a}{2}-\frac{b}{2}\\ 0 & 0 & 0 & -a+b+c\end{bmatrix}$ ← The second row was multiplied by $-\frac{1}{2}$.

If $-a+b+c=0$ then the linear system is consistent. Otherwise (if $-a+b+c\neq 0$) it is inconsistent.

28. $\begin{bmatrix}1 & 3 & 1 & a\\ -1 & -2 & 1 & b\\ 3 & 7 & -1 & c\end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix}1 & 3 & 1 & a\\ 0 & 1 & 2 & a+b\\ 0 & -2 & -4 & -3a+c\end{bmatrix}$ ← The first row was added to the second row and -3 times the first row was added to the third row.

$\begin{bmatrix}1 & 3 & 1 & a\\ 0 & 1 & 2 & a+b\\ 0 & 0 & 0 & -a+2b+c\end{bmatrix}$ ← 2 times the second row was added to the third row.

If $-a+2b+c=0$ then the linear system is consistent. Otherwise (if $-a+2b+c\neq 0$) it is inconsistent.

29. $\begin{bmatrix}2 & 1 & a\\ 3 & 6 & b\end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{2}a\\ 3 & 6 & b\end{bmatrix}$ ← The first row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 0 & \frac{9}{2} & -\frac{3}{2}a+b \end{bmatrix}$ ⟵ -3 times the first row was added to the second row.

$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 0 & 1 & -\frac{1}{3}a+\frac{2}{9}b \end{bmatrix}$ ⟵ The third row was multiplied by $\frac{2}{9}$.

$\begin{bmatrix} 1 & 0 & \frac{2}{3}a-\frac{1}{9}b \\ 0 & 1 & -\frac{1}{3}a+\frac{2}{9}b \end{bmatrix}$ ⟵ $-\frac{1}{2}$ times the second row was added to the first row.

The system has exactly one solution: $x=\frac{2}{3}a-\frac{1}{9}b$ and $y=-\frac{1}{3}a+\frac{2}{9}b$.

30. $\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 0 & -2a+b \\ 0 & 3 & 3 & c \end{bmatrix}$ ⟵ -2 times the first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 3 & 3 & c \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{1}{2}$.

$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 3 & -3a+\frac{3}{2}b+c \end{bmatrix}$ ⟵ -3 times the second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix}$ ⟵ The third row was multiplied by $\frac{1}{3}$.

$\begin{bmatrix} 1 & 1 & 0 & 2a-\frac{b}{2}-\frac{c}{3} \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix}$ ⟵ -1 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & a-\frac{c}{3} \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix}$ ⟵ -1 times the second row was added to the first row.

The system has exactly one solution: $x_1=a-\frac{c}{3}$, $x_2=a-\frac{b}{2}$, and $x_3=-a+\frac{b}{2}+\frac{c}{3}$.

31. Adding -2 times the first row to the second yields a matrix in row echelon form $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

Adding -3 times its second row to the first results in $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, which is also in row echelon form.

32. $\begin{bmatrix}2 & 1 & 3\\ 0 & -2 & -29\\ 3 & 4 & 5\end{bmatrix}$

$\begin{bmatrix}2 & 1 & 3\\ 0 & -2 & -29\\ 1 & 3 & 2\end{bmatrix}$ ⟵ -1 times the first row was added to the third row.

$\begin{bmatrix}1 & 3 & 2\\ 0 & -2 & -29\\ 2 & 1 & 3\end{bmatrix}$ ⟵ The first and third rows were interchanged.

$\begin{bmatrix}1 & 3 & 2\\ 0 & -2 & -29\\ 0 & -5 & -1\end{bmatrix}$ ⟵ -2 times the first row was added to the third row.

$\begin{bmatrix}1 & 3 & 2\\ 0 & -2 & -29\\ 0 & 1 & 86\end{bmatrix}$ ⟵ -3 times the second row was added to the third row.

$\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 86\\ 0 & -2 & -29\end{bmatrix}$ ⟵ The second and third rows were interchanged.

$\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 86\\ 0 & 0 & 143\end{bmatrix}$ ⟵ 2 times the second row was added to the third row.

$\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 86\\ 0 & 0 & 1\end{bmatrix}$ ⟵ The third row was multiplied by $\frac{1}{143}$.

$\begin{bmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ⟵ -86 times the third row was added to the second row and -2 times the third row was added to the first row.

$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ ⟵ -3 times the second row was added to the first row.

33. We begin by substituting $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$ so that the system becomes

$$\begin{aligned} x &+ 2y &+ 3z &= 0 \\ 2x &+ 5y &+ 3z &= 0 \\ -x &- 5y &+ 5z &= 0 \end{aligned}$$

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{bmatrix}$ ← -2 times the first row was added to the second row and the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ ← 3 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The third row was multiplied by -1.

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← 3 times the third row was added to the second row and -3 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← -2 times the second row was added to the first row.

This system has exactly one solution $x = 0,\ y = 0,\ z = 0.$

On the interval $0 \le \alpha \le 2\pi$, the equation $\sin\alpha = 0$ has three solutions: $\alpha = 0$, $\alpha = \pi$, and $\alpha = 2\pi$.

On the interval $0 \le \beta \le 2\pi$, the equation $\cos\beta = 0$ has two solutions: $\beta = \frac{\pi}{2}$ and $\beta = \frac{3\pi}{2}$.

On the interval $0 \le \gamma \le 2\pi$, the equation $\tan\gamma = 0$ has three solutions: $\gamma = 0$, $\gamma = \pi$, and $\gamma = 2\pi$.

Overall, $3 \cdot 2 \cdot 3 = 18$ solutions (α, β, γ) can be obtained by combining the values of α, β, and γ listed above: $\left(0, \frac{\pi}{2}, 0\right), \left(\pi, \frac{\pi}{2}, 0\right)$, etc.

34. We begin by substituting $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$ so that the system becomes

$$\begin{aligned} 2x &- y &+ 3z &= 3 \\ 4x &+ 2y &- 2z &= 2 \\ 6x &- 3y &+ z &= 9 \end{aligned}$$

$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{bmatrix}$ ← -2 times the first row was added to the second row and -3 times the first row was added to the third row.

$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The third row was multiplied by $-\frac{1}{8}$.

$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← 8 times the third row was added to the second row and -3 times the third row was added to the first row.

$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The second row was multiplied by $\frac{1}{4}$.

$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The second row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The first row was multiplied by $\frac{1}{2}$.

This system has exactly one solution $x=1,\ y=-1,\ z=0$.

The only angles $\alpha, \beta,$ and γ that satisfy the inequalities $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, $0 \le \gamma < \pi$ and the equations

$$\sin\alpha = 1, \qquad \cos\beta = -1, \qquad \tan\gamma = 0$$

are $\alpha = \frac{\pi}{2}$, $\beta = \pi$, and $\gamma = 0$.

35. We begin by substituting $X = x^2$, $Y = y^2$, and $Z = z^2$ so that the system becomes

$$\begin{aligned} X + Y + Z &= 6 \\ X - Y + 2Z &= 2 \\ 2X + Y - Z &= 3 \end{aligned}$$

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{bmatrix}$ ← -1 times the first row was added to the second row and -2 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -9 \\ 0 & -2 & 1 & -4 \end{bmatrix}$ ← The second and third rows were interchanged (to avoid introducing fractions into the second row).

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & -2 & 1 & -4 \end{bmatrix}$ ← The second row was multiplied by -1.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 7 & 14 \end{bmatrix}$ ← 2 times the second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← The third row was multiplied by $\frac{1}{7}$.

$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -3 times the third row was added to the second row and -1 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -1 times the second row was added to the first row.

We obtain

$$\begin{aligned} X &= 1 \quad \Rightarrow \quad x = \pm 1 \\ Y &= 3 \quad \Rightarrow \quad y = \pm\sqrt{3} \\ Z &= 2 \quad \Rightarrow \quad z = \pm\sqrt{2} \end{aligned}$$

36. We begin by substituting $a = \frac{1}{x}$, $b = \frac{1}{y}$, and $c = \frac{1}{z}$ so that the system becomes

$$\begin{aligned} a + 2b - 4c &= 1 \\ 2a + 3b + 8c &= 0 \\ -a + 9b + 10c &= 5 \end{aligned}$$

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ -1 & 9 & 10 & 5 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 11 & 6 & 6 \end{bmatrix}$ ← -2 times the first row was added to the second row and the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 11 & 6 & 6 \end{bmatrix}$ ← The second row was multiplied by -1.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 182 & -16 \end{bmatrix}$ ← -11 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & -\frac{8}{91} \end{bmatrix}$ ← The third row was multiplied by $\frac{1}{182}$.

Using back-substitution, we obtain

$$c = -\frac{8}{91} \quad \Rightarrow \quad z = \frac{1}{c} = -\frac{91}{8}$$
$$b = 2 + 16c = \frac{54}{91} \quad \Rightarrow \quad y = \frac{1}{b} = \frac{91}{54}$$
$$a = 1 - 2b + 4c = -\frac{7}{13} \quad \Rightarrow \quad x = \frac{1}{a} = -\frac{13}{7}$$

37. Each point on the curve yields an equation, therefore we have a system of four equations

equation corresponding to $(1,7)$:	$a + b + c + d = 7$
equation corresponding to $(3,-11)$:	$27a + 9b + 3c + d = -11$
equation corresponding to $(4,-14)$:	$64a + 16b + 4c + d = -14$
equation corresponding to $(0,10)$:	$d = 10$

$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 27 & 9 & 3 & 1 & -11 \\ 64 & 16 & 4 & 1 & -14 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -18 & -24 & -26 & -200 \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$ ← -27 times the first row was added to the second row and -64 times the first row was added to the third.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← The second row was multiplied by $-\frac{1}{18}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 4 & \frac{19}{3} & \frac{214}{3} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← 48 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 1 & \frac{19}{12} & \frac{107}{6} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{4}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -3 \\ 0 & 1 & \frac{4}{3} & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← $-\frac{19}{12}$ times the fourth row was added to the third row, $-\frac{13}{9}$ times the fourth row was added to the second row, and -1 times the fourth row was added to the first.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← $-\frac{4}{3}$ times the third row was added to the second row and -1 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← -1 times the second row was added to the first row.

The linear system has a unique solution: $a=1$, $b=-6$, $c=2$, $d=10$. These are the coefficient values required for the curve $y=ax^3+bx^2+cx+d$ to pass through the four given points.

38. Each point on the curve yields an equation, therefore we have a system of three equations

equation corresponding to $(-2,7)$:	$53a - 2b + 7c + d = 0$
equation corresponding to $(-4,5)$:	$41a - 4b + 5c + d = 0$
equation corresponding to $(4,-3)$:	$25a + 4b - 3c + d = 0$

The augmented matrix of this system $\begin{bmatrix} 53 & -2 & 7 & 1 & 0 \\ 41 & -4 & 5 & 1 & 0 \\ 25 & 4 & -3 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{29} & 0 \\ 0 & 1 & 0 & -\frac{2}{29} & 0 \\ 0 & 0 & 1 & -\frac{4}{29} & 0 \end{bmatrix}$$

If we assign d an arbitrary value t, the general solution is given by the formulas

$$a = -\frac{1}{29}t, \quad b = \frac{2}{29}t, \quad c = \frac{4}{29}t, \quad d = t$$

(For instance, letting the free variable d have the value -29 yields $a = 1$, $b = -2$, and $c = -4$.)

39. Since the homogeneous system has only the trivial solution, its augmented matrix must be possible to reduce via a sequence of elementary row operations to the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Applying the **same** sequence of elementary row operations to the augmented matrix of the nonhomogeneous system yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{bmatrix}$ where r, s, and t are some real numbers. Therefore, the nonhomogeneous system has one solution.

40. **(a)** 3 (this will be the number of leading 1's if the matrix has no rows of zeros)

(b) 5 (if all entries in B are 0)

(c) 2 (this will be the number of rows of zeros if each column contains a leading 1)

41. **(a)** There are eight possible reduced row echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where r and s can be any real numbers.

(b) There are sixteen possible reduced row echelon forms:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&r\\0&1&0&s\\0&0&1&t\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&r&0\\0&1&s&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&r&t\\0&1&s&u\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&r&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&r&0&s\\0&0&1&t\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\begin{bmatrix}1&r&s&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&r&s&t\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&r\\0&0&1&s\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&r&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&r&s\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&r\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \text{ and } \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

where r, s, t, and u can be any real numbers.

42. **(a)** Either the three lines properly intersect at the origin, or two of them completely overlap and the other one intersects them at the origin.

(b) All three lines completely overlap one another.

43. **(a)** We consider two possible cases: (i) $a=0$, and (ii) $a \neq 0$.

(i) If $a=0$ then the assumption $ad-bc \neq 0$ implies that $b \neq 0$ and $c \neq 0$. Gauss-Jordan elimination yields

$\begin{bmatrix}0&b\\c&d\end{bmatrix}$ ← We assumed $a=0$

$\begin{bmatrix}c&d\\0&b\end{bmatrix}$ ← The rows were interchanged.

$\begin{bmatrix}1&\frac{d}{c}\\0&1\end{bmatrix}$ ← The first row was multiplied by $\frac{1}{c}$ and the second row was multiplied by $\frac{1}{b}$. (Note that $b,c \neq 0$.)

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ ← $-\frac{d}{c}$ times the second row was added to the first row.

(ii) If $a \neq 0$ then we perform Gauss-Jordan elimination as follows:

$\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$\begin{bmatrix}1&\frac{b}{a}\\c&d\end{bmatrix}$ ← The first row was multiplied by $\frac{1}{a}$.

$\begin{bmatrix}1&\frac{b}{a}\\0&\frac{ad-bc}{a}\end{bmatrix}$ ← $-c$ times the first row was added to the second row.

$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$ ← The second row was multiplied by $\frac{a}{ad-bc}$. (Note that both a and $ad-bc$ are nonzero.)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← $-\frac{b}{a}$ times the second row was added to the first row.

In both cases ($a=0$ as well as $a \neq 0$) we established that the reduced row echelon form of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ provided that $ad-bc \neq 0$.

(b) Applying the **same** elementary row operation steps as in part (a) the augmented matrix $\begin{bmatrix} a & b & k \\ c & d & l \end{bmatrix}$ will be transformed to a matrix in reduced row echelon form $\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}$ where p and q are some real numbers. We conclude that the given linear system has exactly one solution: $x=p$, $y=q$.

True-False Exercises

(a) True. A matrix in reduced row echelon form has all properties required for the row echelon form.

(b) False. For instance, interchanging the rows of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yields a matrix that is not in row echelon form.

(c) False. See Exercise 31.

(d) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. The result follows from Theorem 1.2.1.

(e) True. This is implied by the third property of a row echelon form (see Section 1.2).

(f) False. Nonzero entries are permitted above the leading 1's in a row echelon form.

(g) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. From Theorem 1.2.1 we conclude that the system has $n-n=0$ free variables, i.e. it has only the trivial solution.

(h) False. The row of zeros imposes no restriction on the unknowns and can be omitted. Whether the system has infinitely many, one, or no solution(s) depends *solely* on the nonzero rows of the reduced row echelon form.

(i) False. For example, the following system is clearly inconsistent:

$$\begin{aligned} x+y+z&=1 \\ x+y+z&=2 \end{aligned}$$

1.3 Matrices and Matrix Operations

1. **(a)** Undefined (the number of columns in B does not match the number of rows in A)

(b) Defined; 4×4 matrix

(c) Defined; 4×2 matrix

(d) Defined; 5×2 matrix

(e) Defined; 4×5 matrix

(f) Defined; 5×5 matrix

2. **(a)** Defined; 5×4 matrix

(b) Undefined (the number of columns in D does not match the number of rows in C)

(c) Defined; 4×2 matrix

(d) Defined; 2×4 matrix

(e) Defined; 5×2 matrix

(f) Undefined (BA^T is a 4×4 matrix, which cannot be added to a 4×2 matrix D)

3. **(a)** $$\begin{bmatrix} 1+6 & 5+1 & 2+3 \\ -1+(-1) & 0+1 & 1+2 \\ 3+4 & 2+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1-6 & 5-1 & 2-3 \\ -1-(-1) & 0-1 & 1-2 \\ 3-4 & 2-1 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 5\cdot 3 & 5\cdot 0 \\ 5\cdot(-1) & 5\cdot 2 \\ 5\cdot 1 & 5\cdot 1 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$$

(d) $$\begin{bmatrix} -7\cdot 1 & -7\cdot 4 & -7\cdot 2 \\ -7\cdot 3 & -7\cdot 1 & -7\cdot 5 \end{bmatrix} = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$$

(e) Undefined (a 2×3 matrix C cannot be subtracted from a 2×2 matrix $2B$)

(f) $$\begin{bmatrix} 4\cdot 6 & 4\cdot 1 & 4\cdot 3 \\ 4\cdot(-1) & 4\cdot 1 & 4\cdot 2 \\ 4\cdot 4 & 4\cdot 1 & 4\cdot 3 \end{bmatrix} - \begin{bmatrix} 2\cdot 1 & 2\cdot 5 & 2\cdot 2 \\ 2\cdot(-1) & 2\cdot 0 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 24-2 & 4-10 & 12-4 \\ -4-(-2) & 4-0 & 8-2 \\ 16-6 & 4-4 & 12-8 \end{bmatrix}$$
$$= \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$$

(g) $$-3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2\cdot 6 & 2\cdot 1 & 2\cdot 3 \\ 2\cdot(-1) & 2\cdot 1 & 2\cdot 2 \\ 2\cdot 4 & 2\cdot 1 & 2\cdot 3 \end{bmatrix}\right) = -3\begin{bmatrix} 1+12 & 5+2 & 2+6 \\ -1+(-2) & 0+2 & 1+4 \\ 3+8 & 2+2 & 4+6 \end{bmatrix}$$
$$= \begin{bmatrix} -3\cdot 13 & -3\cdot 7 & -3\cdot 8 \\ -3\cdot(-3) & -3\cdot 2 & -3\cdot 5 \\ -3\cdot 11 & -3\cdot 4 & -3\cdot 10 \end{bmatrix} = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

(h) $\begin{bmatrix} 3-3 & 0-0 \\ -1-(-1) & 2-2 \\ 1-1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) $1+0+4=5$

(j) $\operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 3\cdot 6 & 3\cdot 1 & 3\cdot 3 \\ 3\cdot(-1) & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 1 & 3\cdot 3 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 1-18 & 5-3 & 2-9 \\ -1-(-3) & 0-3 & 1-6 \\ 3-12 & 2-3 & 4-9 \end{bmatrix}\right)$

$= \operatorname{tr}\left(\begin{bmatrix} -17 & 2 & -7 \\ 2 & -3 & -5 \\ -9 & -1 & -5 \end{bmatrix}\right) = -17-3-5 = -25$

(k) $4\operatorname{tr}\left(\begin{bmatrix} 7\cdot 4 & 7\cdot(-1) \\ 7\cdot 0 & 7\cdot 2 \end{bmatrix}\right) = 4\operatorname{tr}\left(\begin{bmatrix} 28 & -7 \\ 0 & 14 \end{bmatrix}\right) = 4(28+14) = 4\cdot 42 = 168$

(l) Undefined (trace is only defined for square matrices)

4. **(a)** $2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 3+1 & 2\cdot(-1)+4 & 2\cdot 1+2 \\ 2\cdot 0+3 & 2\cdot 2+1 & 2\cdot 1+5 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1-6 & -1-(-1) & 3-4 \\ 5-1 & 0-1 & 2-1 \\ 2-3 & 1-2 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(c) $\left(\begin{bmatrix} 1-6 & 5-1 & 2-3 \\ -1-(-1) & 0-1 & 1-2 \\ 3-4 & 2-1 & 4-3 \end{bmatrix}\right)^T = \left(\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(d) Undefined (a 2×2 matrix B^T cannot be added to a 3×2 matrix $5C^T$)

(e) $\begin{bmatrix} \frac{1}{2}\cdot 1 & \frac{1}{2}\cdot 3 \\ \frac{1}{2}\cdot 4 & \frac{1}{2}\cdot 1 \\ \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot 5 \end{bmatrix} - \begin{bmatrix} \frac{1}{4}\cdot 3 & \frac{1}{4}\cdot 0 \\ \frac{1}{4}\cdot(-1) & \frac{1}{4}\cdot 2 \\ \frac{1}{4}\cdot 1 & \frac{1}{4}\cdot 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}-\frac{3}{4} & \frac{3}{2}-0 \\ 2+\frac{1}{4} & \frac{1}{2}-\frac{1}{2} \\ 1-\frac{1}{4} & \frac{5}{2}-\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{9}{4} \end{bmatrix}$

(f) $\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-4 & -1-0 \\ 0-(-1) & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(g) $2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 6 & 2\cdot(-1) & 2\cdot 4 \\ 2\cdot 1 & 2\cdot 1 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 3 \end{bmatrix} - \begin{bmatrix} 3\cdot 1 & 3\cdot(-1) & 3\cdot 3 \\ 3\cdot 5 & 3\cdot 0 & 3\cdot 2 \\ 3\cdot 2 & 3\cdot 1 & 3\cdot 4 \end{bmatrix}$

$= \begin{bmatrix} 12-3 & -2-(-3) & 8-9 \\ 2-15 & 2-0 & 2-6 \\ 6-6 & 4-3 & 6-12 \end{bmatrix} = \begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$

(h) $$\left(2\begin{bmatrix}6 & -1 & 4\\1 & 1 & 1\\3 & 2 & 3\end{bmatrix}-3\begin{bmatrix}1 & -1 & 3\\5 & 0 & 2\\2 & 1 & 4\end{bmatrix}\right)^T=\left(\begin{bmatrix}2\cdot 6 & 2\cdot(-1) & 2\cdot 4\\2\cdot 1 & 2\cdot 1 & 2\cdot 1\\2\cdot 3 & 2\cdot 2 & 2\cdot 3\end{bmatrix}-\begin{bmatrix}3\cdot 1 & 3\cdot(-1) & 3\cdot 3\\3\cdot 5 & 3\cdot 0 & 3\cdot 2\\3\cdot 2 & 3\cdot 1 & 3\cdot 4\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}12-3 & -2-(-3) & 8-9\\2-15 & 2-0 & 2-6\\6-6 & 4-3 & 6-12\end{bmatrix}\right)^T=\left(\begin{bmatrix}9 & 1 & -1\\-13 & 2 & -4\\0 & 1 & -6\end{bmatrix}\right)^T=\begin{bmatrix}9 & -13 & 0\\1 & 2 & 1\\-1 & -4 & -6\end{bmatrix}$$

(i) $$\begin{bmatrix}(1\cdot 1)-(4\cdot 1)+(2\cdot 3) & (1\cdot 5)+(4\cdot 0)+(2\cdot 2) & (1\cdot 2)+(4\cdot 1)+(2\cdot 4)\\(3\cdot 1)-(1\cdot 1)+(5\cdot 3) & (3\cdot 5)+(1\cdot 0)+(5\cdot 2) & (3\cdot 2)+(1\cdot 1)+(5\cdot 4)\end{bmatrix}\begin{bmatrix}6 & 1 & 3\\-1 & 1 & 2\\4 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}3 & 9 & 14\\17 & 25 & 27\end{bmatrix}\begin{bmatrix}6 & 1 & 3\\-1 & 1 & 2\\4 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(3\cdot 6)-(9\cdot 1)+(14\cdot 4) & (3\cdot 1)+(9\cdot 1)+(14\cdot 1) & (3\cdot 3)+(9\cdot 2)+(14\cdot 3)\\(17\cdot 6)-(25\cdot 1)+(27\cdot 4) & (17\cdot 1)+(25\cdot 1)+(27\cdot 1) & (17\cdot 3)+(25\cdot 2)+(27\cdot 3)\end{bmatrix}$$

$$=\begin{bmatrix}65 & 26 & 69\\185 & 69 & 182\end{bmatrix}$$

(j) Undefined (a 2×2 matrix B cannot be multiplied by a 3×2 matrix A)

(k) $$\operatorname{tr}\left(\begin{bmatrix}1 & 5 & 2\\-1 & 0 & 1\\3 & 2 & 4\end{bmatrix}\begin{bmatrix}6 & -1 & 4\\1 & 1 & 1\\3 & 2 & 3\end{bmatrix}\right)$$

$$=\operatorname{tr}\left(\begin{bmatrix}(1\cdot 6)+(5\cdot 1)+(2\cdot 3) & -(1\cdot 1)+(5\cdot 1)+(2\cdot 2) & (1\cdot 4)+(5\cdot 1)+(2\cdot 3)\\-(1\cdot 6)+(0\cdot 1)+(1\cdot 3) & (1\cdot 1)+(0\cdot 1)+(1\cdot 2) & -(1\cdot 4)+(0\cdot 1)+(1\cdot 3)\\(3\cdot 6)+(2\cdot 1)+(4\cdot 3) & -(3\cdot 1)+(2\cdot 1)+(4\cdot 2) & (3\cdot 4)+(2\cdot 1)+(4\cdot 3)\end{bmatrix}\right)$$

$$=\operatorname{tr}\left(\begin{bmatrix}17 & 8 & 15\\-3 & 3 & -1\\32 & 7 & 26\end{bmatrix}\right)=17+3+26=46$$

(l) Undefined (BC is a 2×3 matrix; trace is only defined for square matrices)

5. **(a)** $$\begin{bmatrix}(3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2)\\-(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2)\\(1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2)\end{bmatrix}=\begin{bmatrix}12 & -3\\-4 & 5\\4 & 1\end{bmatrix}$$

(b) Undefined (the number of columns of B does not match the number of rows in A)

(c) $\begin{bmatrix} 3\cdot 6 & 3\cdot 1 & 3\cdot 3 \\ 3\cdot(-1) & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 1 & 3\cdot 3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} (18\cdot 1)-(3\cdot 1)+(9\cdot 3) & (18\cdot 5)+(3\cdot 0)+(9\cdot 2) & (18\cdot 2)+(3\cdot 1)+(9\cdot 4) \\ -(3\cdot 1)-(3\cdot 1)+(6\cdot 3) & -(3\cdot 5)+(3\cdot 0)+(6\cdot 2) & -(3\cdot 2)+(3\cdot 1)+(6\cdot 4) \\ (12\cdot 1)-(3\cdot 1)+(9\cdot 3) & (12\cdot 5)+(3\cdot 0)+(9\cdot 2) & (12\cdot 2)+(3\cdot 1)+(9\cdot 4) \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$$

(d) $\begin{bmatrix} (3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2) \\ -(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2) \\ (1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2) \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} (12\cdot 1)-(3\cdot 3) & (12\cdot 4)-(3\cdot 1) & (12\cdot 2)-(3\cdot 5) \\ -(4\cdot 1)+(5\cdot 3) & -(4\cdot 4)+(5\cdot 1) & -(4\cdot 2)+(5\cdot 5) \\ (4\cdot 1)+(1\cdot 3) & (4\cdot 4)+(1\cdot 1) & (4\cdot 2)+(1\cdot 5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(e) $\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (4\cdot 1)-(1\cdot 3) & (4\cdot 4)-(1\cdot 1) & (4\cdot 2)-(1\cdot 5) \\ (0\cdot 1)+(2\cdot 3) & (0\cdot 4)+(2\cdot 1) & (0\cdot 2)+(2\cdot 5) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$

$$= \begin{bmatrix} (3\cdot 1)+(0\cdot 6) & (3\cdot 15)+(0\cdot 2) & (3\cdot 3)+(0\cdot 10) \\ -(1\cdot 1)+(2\cdot 6) & -(1\cdot 15)+(2\cdot 2) & -(1\cdot 3)+(2\cdot 10) \\ (1\cdot 1)+(1\cdot 6) & (1\cdot 15)+(1\cdot 2) & (1\cdot 3)+(1\cdot 10) \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(f) $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} (1\cdot 1)+(4\cdot 4)+(2\cdot 2) & (1\cdot 3)+(4\cdot 1)+(2\cdot 5) \\ (3\cdot 1)+(1\cdot 4)+(5\cdot 2) & (3\cdot 3)+(1\cdot 1)+(5\cdot 5) \end{bmatrix} = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$

(g) $\left(\begin{bmatrix} (1\cdot 3)-(5\cdot 1)+(2\cdot 1) & (1\cdot 0)+(5\cdot 2)+(2\cdot 1) \\ -(1\cdot 3)-(0\cdot 1)+(1\cdot 1) & -(1\cdot 0)+(0\cdot 2)+(1\cdot 1) \\ (3\cdot 3)-(2\cdot 1)+(4\cdot 1) & (3\cdot 0)+(2\cdot 2)+(4\cdot 1) \end{bmatrix}\right)^T = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$