Chapter 1

Number Systems

1.1 The Real Numbers

1. The set $(0, 1]$ contains its least upper bound 1 but not its greatest lower bound 0. The set $[0, 1)$ contains its greatest lower bound 0 but not its least upper bound 1.

2. The set $\mathbb{Z} \subseteq \mathbb{R}$ has neither a least upper bound nor a greatest lower bound.

3. We know that $\alpha \geq a$ for every element $a \in A$. Thus $-\alpha \leq -a$ for every element $a \in A$ hence $-\alpha \leq b$ for every $b \in B$. If $b' > -\alpha$ is a lower bound for B then $-b' < \alpha$ is an upper bound for A, and that is impossible. Hence $-\alpha$ is the greatest lower bound for B.

Likewise, suppose that β is a greatest lower bound for A. Define $B = \{-a : a \in A\}$. We know that $\beta \leq a$ for every element $a \in A$. Thus $-\beta \geq -a$ for every element $a \in A$ hence $-\beta \geq b$ for every $b \in B$. If $b' < -\beta$ is an upper bound for B then $-b' > \beta$ is a lower bound for A, and that is impossible. Hence $-\beta$ is the least upper bound for B.

4. The least upper bound for S is $\sqrt{2}$.

5. We shall treat the least upper bound. Let α be the least upper bound for the set S. Suppose that α' is another least upper bound. It $\alpha' > \alpha$ then α' cannot be the least upper bound. If $\alpha' < \alpha$ then α cannot be the least upper bound. So α' must equal α.

6. Certainly S is bounded above by the circumference of C. The least upper bound of S is π. This exercise cannot work in the rational number system because π is irrational.

7. Let x and y be real numbers. We know that

$$(x+y)^2 = x^2 + 2xy + y^2 \le |x|^2 + 2|x||y| + |y|^2 .$$

Taking square roots of both sides yields

$$|x+y| \le |x| + |y| .$$

8. We treat the supremum. Notice that, since the empty set has no elements, then $-\infty \ge x$ for all $x \in \emptyset$ vacuously. There are no real numbers less than $-\infty$, so $-\infty$ is the supremum of $\emptyset$.

9. We treat commutativity. According to the definition in the text, we add two cuts $\mathcal{C}$ and $\mathcal{D}$ by

$$\mathcal{C} + \mathcal{D} = \{c + d : c \in \mathcal{C}, d \in \mathcal{D}\} .$$

But this equals

$$\{d + c : c \in \mathcal{C}, d \in \mathcal{D}\}$$

and that equals $\mathcal{D} + \mathcal{C}$.

11. Consider the set of all numbers of the form

$$\frac{j}{k\sqrt{2}}$$

for j, k relatively prime natural numbers and $j < k$. Then certainly each of these numbers lies between 0 and 1 and each is irrational. Furthermore, there are countably many of them.

* **12.** Let x be in the domain of f. Then x is a local minimum, so there are rational numbers $\alpha_x < x < \beta_x$ so that

$$f(x) \le f(t)$$

for every $t \in (\alpha_x, \beta_x)$. Thus we associate to each value $f(x)$ of the function f a pair of rational numbers (α_x, β_x). But the set of such pairs is countable. So the set of values of f is countable.

* **13.** Notice that if $n - k\lambda = m - \ell\lambda$ then $(n-m) = (k-\ell)\lambda$. It would follow that λ is rational unless $n = m$ and $k = \ell$. So the numbers $n - k\lambda$ are all distinct.

Now let $\epsilon > 0$ and choose an positive integer N so large that $\lambda/N < \epsilon$. Consider $\varphi(1)$, $\varphi(2)$, $\dots$, $\varphi(N)$. These numbers are all distinct, and lie in the interval $[0, \lambda]$. So two of them are distance not more than $\lambda/N < \epsilon$ apart. Thus $|(n_1 - k_1\lambda) - (n_2 - k_2\lambda)| < \epsilon$ or $|(n_1 - n_2) - (k_1 - k_2)\lambda| < \epsilon$. Let us abbreviate this as $|m - p\lambda| < \epsilon$.

It follows then that the numbers

$$(m - p\lambda), (2m - 2p\lambda), (3m - 3p\lambda), \dots$$

are less than ϵ apart and fill up the interval $[0, \lambda]$. That is the definition of density.

1.2 The Complex Numbers

1. We calculate that

$$z \cdot \frac{\overline{z}}{|z|^2} = \frac{z \cdot \overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1\,.$$

So $\overline{z}/|z|^2$ is the multiplicative inverse of z.

2. We calculate that

$$\overline{z/w} = \overline{z \cdot \frac{1}{w}} = \overline{z \cdot \frac{\overline{w}}{|w|^2}} = \overline{z} \cdot \overline{\overline{w}/|w|^2} = \overline{z} \cdot \frac{w}{|w|^2} = \frac{\overline{z}}{\overline{w}}\,.$$

3. Write

$$1 + i = \sqrt{2}e^{i\pi/4}\,.$$

We seek a complex number $z = re^{i\theta}$ such that

$$z^3 = r^3 e^{3i\theta} = (re^{i\theta})^3 = \sqrt{2}e^{i\pi/4}\,.$$

It follows that $r = 2^{1/6}$ and $\theta = \pi/12$. So we have found the cube root

$$c_1 = 2^{1/6}e^{i\pi/12}\,.$$

Now we may repeat this process with $\sqrt{2}e^{i\pi/4}$ replaced by $\sqrt{2}e^{i9\pi/4}$. We find the second cube root

$$c_2 = 2^{1/6}e^{i9\pi/12} .$$

Repeating the process a third time with $\sqrt{2}e^{i\pi/4}$ replaced by $\sqrt{2}e^{i17\pi/4}$, we find the third cube root

$$c_3 = 2^{1/6}e^{i17\pi/12} .$$

4. We first treat the commutative law for addition. Let $z = x + iy$ and $w = u + iv$. Then

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v) .$$

Now we invoke the commutative law of addition for the real numbers to write this as

$$(u + x) + i(v + y) = (u + iv) + (x + iy) = w + z .$$

Now let us treat the commutative law for multiplication. With z, w as above, we write

$$z \cdot w = (x + iy) \cdot (u + iv) = (xu - yv) + i(xv + uy) .$$

Now we invoke the commutative law for multiplication of real numbers, as well as the commutative law for addition of real numbers, to rewrite this as

$$(ux - vy) + i(uy + xv) = (u + iv) \cdot (x + iy) = w \cdot z .$$

5. We see that

$$\phi(x + x') = (x + x') + i0 = (x + i0) + (x' + i0) = \phi(x) + \phi(x') .$$

Also

$$\phi(x \cdot x') = (x \cdot x') + i0 = (x + i0) \cdot (x' + i0) = \phi(x) \cdot \phi(x') .$$

6. Since $i \neq 0$, then either $i > 0$ or $i < 0$. If $i > 0$, then $i \cdot i > 0$. But $i \cdot i = -1 < 0$. Contradiction. If instead $i < 0$, then $-i > 0$. Hence $(-i) \cdot (-i) > 0$. Thus again $-1 > 0$ and that is false. So we see that the complex numbers cannot be ordered.

7. Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$$

be a polynomial with real coefficients a_j. If α is a root of this polynomial then

$$p(\alpha) = a_0 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_k \alpha^k = 0\,.$$

Conjugating this equation gives

$$p(\alpha) = a_0 + a_1 \overline{\alpha} + a_2 \overline{\alpha}^2 + \cdots + a_k \overline{\alpha}^k = 0\,.$$

Hence $\overline{\alpha}$ is a root of the polynomial p. We see then that roots of p occur in conjugate pairs.

8. Write

$$i = 1 \cdot e^{i\pi/2}\,.$$

We seek a complex number $z = re^{i\theta}$ so that $z^2 = 1 \cdot e^{i\pi/2}$. Thus

$$r^2 e^{2i\theta} = 1 \cdot e^{i\pi/2}\,.$$

We conclude that $r = 1$ and $\theta = \pi/4$. So we have found the square root

$$c_1 = 1 \cdot e^{i\pi/4}\,.$$

We may repeat this construction with $1 \cdot e^{i\pi/2}$ replaced by $1 \cdot e^{i5\pi/2}$. We find the second square root

$$c_2 = 1 \cdot e^{5\pi/4}\,.$$

9. The function $\varphi(x) = x + i0$ from $\mathbb{R}$ to $\mathbb{C}$ is one-to-one. Therefore

$$\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{C})\,.$$

Since the reals are uncountable, we may conclude that the complex numbers are uncountable.

10. The kth roots of the complex number $\alpha \neq 0$ are the roots of the polynomial $p(z) = z^k - \alpha$. A polynomial of degree k has k roots. Since $p' \neq 0$ except at $z = 0$, we know that these roots are distinct.

11. The defining condition measures the sum of the distance of z to $1 + i0$ plus the distance of z to $-1 + i0$. If z is not on the x-axis then $|z - 1| + |z + 1| > 2$ (by the triangle inequality). If z is on the x axis but less than -1 or greater than 1 then $|z-1|+|z+1| > 2$. So the only z that satisfy $|z - 1| + |z + 1| > 2$ are those elements of the x-axis that are between -1 and 1 inclusive.

12. The k roots of $z = re^{i\theta}$ are the k complex numbers

$$c_j = r^{1/k} e^{i(\theta+2j\pi)/k} \ , \quad 0 \leq j \leq k-1 \, . \qquad (*)$$

We see that these numbers are distinct, and there are k of them. They all have modulus $r^{1/k}$, so they all lie on a circle centered at the origin with radius $r^{1/k}$. The $j$the and $(j + 1)$th points specified in line $(*)$ differ in argument by $2\pi/k$. So they are equally spaced.

14. We write

$$-1 - i = \sqrt{2} \cdot e^{i5\pi/4} \, .$$

We seek a complex number $z = re^{i\theta}$ so that

$$z^2 = r^2 e^{i2\theta} = \sqrt{2} e^{i5\pi/4} \, .$$

Therefore $r = 2^{1/4}$ and $\theta = 5\pi/8$. We have found the square root

$$c_1 = 2^{1/4} e^{i5\pi/8} \, .$$

Now replacing $\sqrt{2}e^{i5\pi/4}$ with $\sqrt{2}e^{i13\pi/4}$, we find a second square root of the form

$$c_2 = 2^{1/4} e^{i13\pi/8} \, .$$

15. The set of all complex numbers with rational real part contains the set of all complex numbers of the form $0 + yi$, where y is any real number. This latter set is plainly uncountable, so the set of complex number with rational real part is also uncountable.

17. The set $S = \{z \in \mathbb{C} : |z| = 1\}$ can be identified with $T = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. The set T can be identified with the interval $[0, 2\pi)$, and that interval is certainly an uncountable set. Hence S is uncountable.

19. Let p be a polynomial of degree $k \geq 1$ and let α_1 be a root of p. So $p(\alpha) = 0$. Now let us think about dividing $p(z)$ by $(z - \alpha_1)$. By the Euclidean algorithm,

$$p(z) = (z - \alpha_1) \cdot q_1(z) + r_1(z)\,. \qquad (*)$$

Here q_1 is the "quotient" and r_1 is the "remainder." The quotient will have degree $k - 1$ and the remainder will have degree *less* than the degree of $z - \alpha_1$. In other words, the remainder will have degree 0—which means that it is constant. Plug the value $z = \alpha_1$ into the equation $(*)$. We obtain

$$0 = 0 + r_1\,.$$

Hence the remainder, the constant r_1, is 0.

If $k = 1$ then the process stops here. If $k > 1$ then q_1 has degree $k - 1 \geq 1$ and we may apply the Fundamental Theorem of Algebra to q_1 to find a root α_2. Repeating the argument above, we divide $(z - \alpha_2)$ into q_1 using the Euclidean algorithm. We find that it divides in evenly, producing a new quotient q_2.

This process can be repeated $k - 2$ more times to produce a total of k roots of the polynomial p.