2 Some Properties of Groups

2.1 Suppose that there are two group identities, e and e'. Then $e = e \cdot e' = e'$. Thus there is only one group identity. Suppose that a^{-1} and $\bar{a}$ are both inverses of the group element a. Then using that $a \cdot a^{-1}$ and $a \cdot \bar{a}$ must both be equal to the group identity,

$$\bar{a} = \bar{a} \cdot (a \cdot a^{-1}) = (\bar{a} \cdot a) \cdot a^{-1} = ea^{-1} = a^{-1}.$$

Thus each group element has a unique inverse. From the properties of the inverse and identity, $(a \cdot b)^{-1} \cdot (a \cdot b) = e$. Multiply this expression from the right by b^{-1} and use $b \cdot b^{-1} = e$ to give

$$(a \cdot b)^{-1} \cdot a = b^{-1}.$$

Multiply from the right by a^{-1} and use $a \cdot a^{-1} = e$ to give

$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}.$$

Check this result against entries in Table 2.2: Let $a = (12)$ and $b = (23)$. Then from the table $a^{-1} = (12)$ and $b^{-1} = (23)$, and also from the table

$$a \cdot b = (12) \cdot (23) = (321).$$

Thus from the table $(a \cdot b)^{-1} = (321)^{-1} = (123)$. From the formula derived above and the table

$$(a \cdot b)^{-1} = b^{-1} a^{-1} = (23) \cdot (12) = (123).$$

Hence for this example $(a \cdot b)^{-1} = b^{-1} a^{-1}$, as asserted.

2.2 Define the permutations

$$(\,) = \begin{pmatrix} 123 \\ 123 \end{pmatrix} \qquad (12) = \begin{pmatrix} 123 \\ 213 \end{pmatrix} \qquad (23) = \begin{pmatrix} 123 \\ 132 \end{pmatrix}$$

$$(13) = \begin{pmatrix} 123 \\ 321 \end{pmatrix} \qquad (123) = \begin{pmatrix} 123 \\ 231 \end{pmatrix} \qquad (321) = \begin{pmatrix} 123 \\ 312 \end{pmatrix}$$

so that, for example, $(12)\{abc\} \to \{bac\}$. Carrying out all possible combinations, the multiplication table for $A \cdot B$ is

$A \backslash B$	e	(12)	(23)	(13)	(123)	(321)
e	e	(12)	(23)	(13)	(123)	(321)
(12)	(12)	e	(321)	(123)	(13)	(23)
(23)	(23)	(123)	e	(321)	(12)	(13)
(13)	(13)	(321)	(123)	e	(23)	(12)
(123)	(123)	(23)	(13)	(12)	(321)	e
(321)	(321)	(13)	(12)	(23)	e	(123)

where e is the identity, $e \equiv (\ )$. This is equivalent to Table 2.2. From the multiplication table this is a group, because all group postulates are satisfied. The identity and any of the three transpositions of two objects are closed under multiplication, so these form 2-element subgroups isomorphic to S_2,

$$S_2 \sim \{e,(12)\} \sim \{e,(13)\} \sim \{e,(23)\}.$$

The set of even permutations $A_3 = \{e,(123),(321)\}$ is closed under multiplication, so this too is a subgroup. The multiplication table for C_3 can be deduced from geometry. The operators 1, c_3, c_3^2, and c_3^3, rotate by 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and $2\pi = 0$, respectively, so the C_3 multiplication table and isomorphism with A_3 are

C_3	e	c_3	c_3^2
e	e	c_3	c_3^2
c_3	c_3	c_3^2	e
c_3^2	c_3^2	e	c_3

$\longleftrightarrow$

A_3	e	(123)	(321)
e	e	(123)	(321)
(123)	(123)	(321)	e
(321)	(321)	e	(123)

if we invoke the correspondence

$$e \longleftrightarrow e \qquad c_3 \longleftrightarrow (123) \qquad c_3^2 \longleftrightarrow (321)$$

between entries in the two multiplication tables.

2.3 The multiplication table is

C_4	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

where $a = c_4$, $a^2 = c_4^2$, $a^3 = c_4^3$, and $a^4 = c_4^4 = e$. Therefore, this is an abelian group.

2.4 Consider the multiplication table for $G = \{e,a,b\}$, where e is the identity. We must have $a \cdot a = b$, since

- If $a \cdot a = a$, then $a = e$ and a group can't have two identities.
- If $a \cdot a = e$, then the elements $\{e,a\}$ alone close a group of order two.

Then since $a \cdot a = b$, it follows that $a \cdot b = b \cdot a = e$ and $b \cdot b = a$ are the only consistent choices. Therefore the multiplication table for the finite group of order three is unique and given by

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

.

We conclude that there is only one independent group of order three. Note, for example, that this multiplication table is isomorphic to that for the cyclic group C_3 if we identify $a = c_3$ and $b = c_3^2$ (see Example 2.18 and Problem 2.2).

2.5 From Example 2.15, the multiplication table for the cyclic group C_4 is

C_4	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

For the subgroup $H = \{e, a^2\}$

$$eee^{-1} = e \qquad aea^{-1} = aa^{-1} = e$$
$$a^2e(a^2)^{-1} = a^2ea^2 = e \qquad a^3e(a^3)^{-1} = a^3ea = e$$
$$ea^2e^{-1} = a^2 \qquad aa^2a^{-1} = aa^2a^3 = a^2$$
$$a^2a^2(a^2)^{-1} = a^2a^2a^2 = a^2 \qquad a^3a^2(a^3)^{-1} = a^3a^2a = a^2,$$

where the cyclic condition $a^4 = e$ has been used and the inverses have been inferred from the multiplication table and the requirement $xx^{-1} = e$. Therefore H is identical to all of its conjugate subgroups

$$H' = \{ghg^{-1};\ g \in G,\ h \in H\},$$

and is an invariant subgroup (Section 2.12). In fact, since all cyclic groups are abelian, it is an abelian invariant subgroup.

2.6 An invariant subgroup consists of whole classes. For S_3 there are three classes:

$$\{e\} \quad \{(12), (23), (13)\} \quad \{(123), (321)\},$$

as may be verified by similarity transforms [see Eq. (2.21)]. Therefore, $\{e, (123), (321)\}$ (the *alternating subgroup*) is invariant because it consists of two whole classes. The subgroup $\{e, (12)\}$ cannot be invariant because e is a whole class but (12) is only part of a class.

2.7 C_4 is abelian (see Problem 2.3). For abelian groups every element is in a class by itself, so *every* subgroup is invariant (consists of whole classes). Therefore, the subgroup $\{e, a^2\}$ is abelian invariant (see Problem 2.5) and C_4 is not simple (it contains invariant subgroups) and it is not semisimple (it contains an abelian invariant subgroup).

2.8 (a) $a \sim a$ requires that there be some group element g such that $a = gag^{-1}$. The choice $g = e$ obviously satisfies this. (b) If $a \sim b$, then $a = gbg^{-1}$ where $g \in G$. Multiply from the left by g^{-1} and from the right by g to give $g^{-1}ag = b$, which implies that $b \sim a$. (c) If $a \sim b$ and $b \sim c$, then there are elements $p, q \in G$ such that

$$a = pbp^{-1} \qquad b = qcq^{-1}.$$

Therefore,

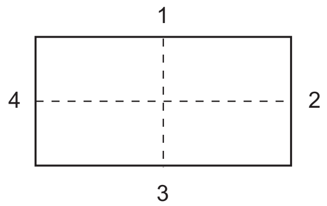
$$a = pbp^{-1} = pqcq^{-1}p^{-1} = pqc(pq)^{-1},$$

where we have used that the reciprocal of a matrix product is formed by taking reciprocals of the factors in reverse order:

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}.$$

Then, since pq must be a group element, $a \sim c$. Thus class conjugation is an equivalence relation, as defined in Box 2.5.

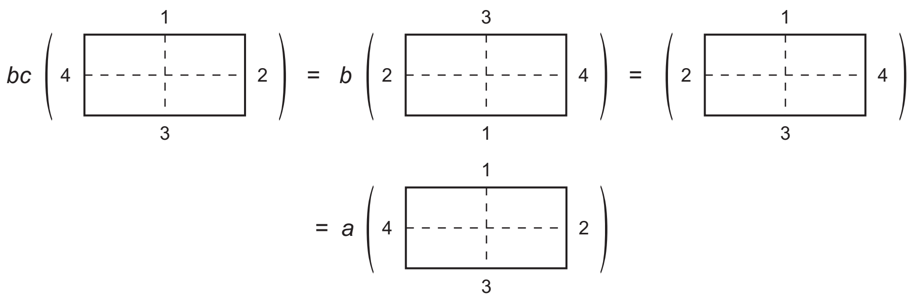
2.9 The figure



has the geometrical symmetry operations:

$e =$ identity
$a =$ reflection through the vertical 1–3 axis
$b =$ reflection through the horizontal 2–4 axis
$c =$ rotation in plane of rectangle about center by π

We may work out the multiplication table by applying all possible pairs of these symmetry operations to the figure. For example, applying the product bc (rotation in the plane by π and then reflection through the horizontal axis),

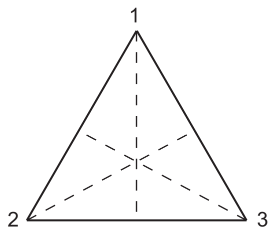


and $bc = a$. Evaluating all such products, the resulting multiplication table is

D_2	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This group is called the *4-group* or *dihedral group* D_2. Generally the dihedral groups D_n correspond to the rotation and reflection symmetries of the regular polygons. From the multiplication table $\{e,a\}$, $\{e,b\}$, and $\{e,c\}$ are subgroups. They are isomorphic to $C_2 \sim Z_2$ because there is only one 2-element group $C_2 \sim Z_2$ (see Box 2.2).

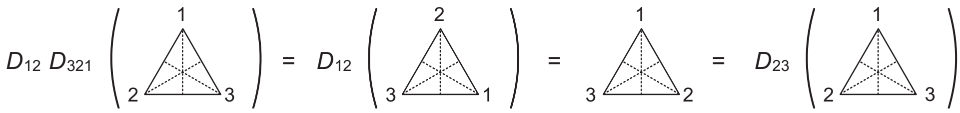
2.10 From the geometry of the figure



a multiplication table may be constructed by applying two successive operations. Let

$$D_{12} = \text{reflection about 3-axis} \qquad e = \text{identity}$$
$$D_{13} = \text{reflection about 2-axis} \qquad D_{321} = \text{clockwise rotation by } 2\pi/3$$
$$D_{23} = \text{reflection about 1-axis} \qquad D_{123} = \text{clockwise rotation by } 4\pi/3.$$

Then, for example, the product $D_{12}D_{321}$ is



and $D_{12}D_{321} = D_{23}$. By carrying out all such pairs of operations, a 6×6 multiplication table may be constructed that is closed (group property) and equivalent to the table of Problem 2.2, with the identification $D_{pq} \to (pq)$. Thus the group is isomorphic to S_3. We have already seen in Problem 2.2 that S_3 has four proper, distinct subgroups: the alternating group, and three S_2 subgroups formed from the identity and one of the three independent transpositions.

2.11 The multiplication table of D_2 is given in Problem 2.9. It has abelian invariant subgroups $\{e,a\}$, $\{e,b\}$, and $\{e,c\}$. Let's form the left cosets of $H = \{e,a\}$:

$$e\{e,a\} = \{e,a\} = H \qquad a\{e,a\} = \{a,e\} = \{e,a\} = H$$
$$b\{e,a\} = \{b,c\} \qquad c\{e,a\} = \{c,b\} = \{b,c\}.$$

Therefore,

$$G/H = D_2/H = \{e,a\} + \{b,c\}.$$

Now let's construct the multiplication table for G/H. Let

$$E = H = \{e,a\} \qquad M = \{b,c\} = b\{e,a\}.$$

From the multiplication table and the coset multiplication law of Eq. (2.27),

$$pH \cdot qH = (pq)H$$

we find that

$$E \cdot E = e\{e,a\}e\{e,a\} = e^2\{e,a\} = E,$$
$$E \cdot M = e\{e,a\}b\{e,a\} = eb\{e,a\} = b\{e,a\} = \{b,c\} = M,$$
$$M \cdot E = b\{e,a\}e\{e,a\} = be\{e,a\} = b\{e,a\} = M,$$
$$M \cdot M = b\{e,a\}b\{e,a\} = bb\{e,a\} = e\{e,a\} = \{e,a\} = E,$$

so the multiplication table of the quotient group is

G/H	E	M
E	E	M
M	M	E

where $E = H$. This is the multiplication table for the group $C_2 \sim Z_2$.

2.12 Consider an operator $U(a)$ that rotates by an infinitesimal amount a around a specified axis. Applying to a wavefunction $\psi(\theta)$,

$$U(a)\psi(\theta) = \psi(\theta + a).$$

Expand in a Taylor series about θ:

$$\begin{aligned}\psi(\theta + a) &= \psi(\theta) + a\frac{d\psi(\theta)}{d\theta} + \frac{a^2}{2!}\frac{d^2\psi(\theta)}{d\theta^2} + \dots \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{d\theta^n}\psi(\theta) = e^{a(d/d\theta)}\psi(\theta) \\ &= e^{ia(-id/d\theta)}\psi(\theta) = e^{\frac{i}{\hbar}aL}\psi(\theta),\end{aligned}$$

where $L \equiv \frac{\hbar}{i} d/d\theta$ is the generator of infinitesimal rotations about this axis,

$$U(a) \simeq 1 + \frac{i}{\hbar}aL.$$

But since the rotations form a continuous and analytical group, all finite rotations connected continuously to the identity can be generated by successive applications of the infinitesimal generator L.

2.13 This problem is adapted from a discussion in O'Raifeartaigh [158]. For the set of upper-triangular 3×3 matrices

$$G = \begin{pmatrix} 1 & \alpha & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

where α, β, and δ are real numbers, a general multiplication of set elements is

$$\begin{pmatrix} 1 & \alpha & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha' & \delta' \\ 0 & 1 & \beta' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha'' & \delta'' \\ 0 & 1 & \beta'' \\ 0 & 0 & 1 \end{pmatrix}$$

where we define

$$\alpha'' \equiv \alpha + \alpha' \qquad \delta'' \equiv \delta + \delta' + \alpha\beta' \qquad \beta'' \equiv \beta + \beta'.$$

Thus these matrices form a group under matrix multiplication (it is called the *Heisenberg group*). Consider the subset of matrices from this group

$$H = a(\delta) = \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

formed by restricting G to those matrices with $\alpha = \beta = 0$. Multiplication of any two such elements gives

$$a(\delta)a(\delta') = \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \delta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta'' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = a(\delta''),$$

where $\delta'' = \delta + \delta'$. Hence the matrices with $\alpha = \beta = 0$ form a subgroup H, which is abelian since $[a(\delta), a(\delta')] = 0$. Forming the cosets for $g \in G$,

$$gH = \begin{pmatrix} 1 & \alpha & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \delta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha & \delta'' \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

$$Hg = \begin{pmatrix} 1 & 0 & \delta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha & \delta'' \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

where $\delta'' \equiv \delta + \delta'$. Thus $Hg = gH$ and H is an (abelian) invariant subgroup of G (see Section 2.12). More formally, we can show by standard matrix inversion that

$$G^{-1} = \begin{pmatrix} 1 & \alpha & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\alpha & \alpha\beta - \delta \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix},$$

so that

$$GHG^{-1} = \begin{pmatrix} 1 & \alpha & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \delta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha & \alpha\beta - \delta \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & \delta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

so H is an invariant subgroup of G, for which left and right cosets are equal. By direct matrix multiplication it is clear that H is abelian, so it is an abelian invariant subgroup.

2.14 This proof follows Elliott and Dawber [56]. We have from Eq. (2.28) and Example 2.19,

$$A^{(\alpha\times\beta)}_{ij,k\ell}(G_a) \equiv A^{(\alpha)}_{ik}(G_a)A^{(\beta)}_{j\ell}(G_a),$$

for the elements of a direct product matrix. Then under regular matrix multiplication of two such matrices that are themselves direct products, the elements are

$$
\begin{aligned}
[A^{(\alpha\times\beta)}(G_a)A^{(\alpha\times\beta)}(G_b)]_{ij,k\ell} &= \sum_{mn} A^{(\alpha\times\beta)}_{ij,mn}(G_a)A^{(\alpha\times\beta)}_{mn,k\ell}(G_b) \\
&= \sum_{mn} A^{(\alpha)}_{im}(G_a)A^{(\beta)}_{jn}(G_a)A^{(\alpha)}_{mk}(G_b)A^{(\beta)}_{n\ell}(G_b) \\
&= \sum_{m} A^{(\alpha)}_{im}(G_a)A^{(\alpha)}_{mk}(G_b)\sum_{n} A^{(\beta)}_{jn}(G_a)A^{(\beta)}_{n\ell}(G_b) \\
&= A^{(\alpha)}_{ik}(G_aG_b)A^{(\beta)}_{j\ell}(G_aG_b) \\
&= A^{(\alpha\times\beta)}_{ij,k\ell}(G_aG_b),
\end{aligned}
$$

where we have used Eq. (2.8) for the original matrices,

$$T(G_a)\cdot T(G_b) = T(G_aG_b).$$

Therefore, the representation condition (2.8) is fulfilled for direct product matrices: if A and B are representations, then their direct product $A\otimes B$ is also a representation. For the characters,

$$
\begin{aligned}
\chi^{(\alpha\times\beta)}(G_a) &= \sum_{ij} A^{(\alpha\times\beta)}_{ij,ij}(G_a) \\
&= \sum_{ij} A^{(\alpha)}_{ii}(G_a)T^{(\beta)}_{jj}(G_a) \\
&= \chi^{(\alpha)}(G_a)\chi^{(\beta)}(G_a).
\end{aligned}
$$

Therefore, the character of a direct product of two representations is the product of characters for the two representations.

2.15 The representations $\Gamma^{(4)}$ and $\Gamma^{(5)}$ are equivalent (they have the same characters), so they are related by a similarity transform. You can verify that $\Gamma^{(5)}$ is converted to $\Gamma^{(4)}$ if each matrix is similarity transformed using [180]

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 1 & \sqrt{3} & -1 \\ 1 & -\sqrt{3} & -1 \end{pmatrix}.$$

Thus it is sufficient to examine one of $\Gamma^{(4)}$ or $\Gamma^{(5)}$. From Eq. (2.32),

$$a_\nu = \frac{1}{N_{\rm G}}\sum_i n_i \chi^*_\nu(i)\chi(i),$$

where the sum is over classes, and $\chi_\nu(i)$ and $\chi(i)$ are the characters of a given class i in the irrep ν and in the reducible representation $U(g)$, respectively. Thus, for $\Gamma^{(4)}$

$$
\begin{aligned}
a_1 &= \frac{1}{6}[(1)(1)(3)+(2)(1)(0)+(3)(1)(1)] = 1, \\
a_2 &= \frac{1}{6}[(1)(1)(3)+(2)(1)(0)+(3)(-1)(1)] = 0, \\
a_3 &= \frac{1}{6}[(1)(2)(3)+(2)(-1)(0)+(3)(0)(1)] = 1,
\end{aligned}
$$

so the $\Gamma^{(4)}$ and $\Gamma^{(5)}$ representations of S_3 displayed in Fig. 2.4 are reducible, with the irrep content $\Gamma^{(1)} \oplus \Gamma^{(3)}$.

2.16 From Eq. (2.27), define coset multiplication by $pHqH = (pq)H$. Then,

1. For H an invariant subgroup, this is a consistent definition and leads to *closure* because $Hg = gH$ and

$$pHqH = pHHq = pHq = pqH,$$

   where we have used that under this multiplication law $eHeH = eeH = H$, so $HH = H$. Therefore, the product of two cosets under coset multiplication is itself a coset, since if $p \in G$ and $q \in G$, then $pq \in G$.
2. Coset multiplication is *associative* since the original group multiplication is associative:

$$\begin{aligned}(pH)\big((qH)(rH)\big) &= (pH)(qrHH)\\ &= p(qr)HHH\\ &= (pq)rHHH\\ &= \big((pH)(qH)\big)(rH).\end{aligned}$$

3. The coset $E \equiv eH = H$ acts as an *identity* since, by the coset multiplication law,

$$eHqH = eqH = qH.$$

4. For each coset pH there is a *unique inverse* $p^{-1}H$, since

$$(pH)(p^{-1}H) = (pp^{-1})H = eH = E.$$

Therefore, the cosets form a group under the coset multiplication law if H is an invariant subgroup.

2.17 If g_iH and g_jH have an element in common, then $g_ih_1 = g_jh_2$ for some elements $h_1, h_2 \in H$, which implies that

$$g_ig_j^{-1} = h_2h_1^{-1}.$$

But $h_2h_1^{-1}$ is an element of H by the group property and, by the rearrangement lemma (Section 6.2.5), $Hg_ig_j^{-1} = H$ and thus $Hg_i = Hg_j$. Thus, two cosets are identical if a common element exists, so different cosets are completely disjoint.

2.18 (a) Let the order of G be n, the order of H be m, with the index of H in G being ℓ, with $n = m\ell$. Then trivially, if n is a prime number either $m = 1$ or $m = n$, so there are no proper subgroups.

(b) Let G be of order n, where n is a prime number, and choose $g \in G$ but not equal to the identity e. Then if we take successive products of g with itself k times, $g^k = e$ by the group property, for some integer k. This means that

$$\left\{g, g^2, \dots, g^{k-1}, e\right\}$$

is a cyclic subgroup. But from part (a) G can't a proper subgroup because it is of prime order by hypothesis. Thus the cyclic subgroup can only be the full group and generally groups of prime order n are isomorphic to cyclic groups C_n.

2.19 Since a is the unit matrix, it will play the role of the identity e. Construct the multiplication table by taking all possible matrix products. For example,

$$b \cdot c = c \cdot b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = d.$$

The resulting multiplication table is

D_2	e	b	c	d
e	e	b	c	d
b	b	e	d	c
c	c	d	e	b
d	d	c	b	e

with $a \equiv e$. This is the multiplication table for the group D_2 (see the solution of Problem 2.9). The group is abelian since the multiplication table is symmetric with respect to reflection through the diagonal.

2.20 Consider the function set

$$f_1(x) = x \qquad f_2(x) = -x \qquad f_3(x) = \frac{1}{x} \qquad f_4(x) = -\frac{1}{x}$$

under the binary operation of substituting one function in another. For example,

$$f_1 \cdot f_1 \equiv f_1(f_1) = x = f_1 \qquad f_1 \cdot f_2 \equiv f_1(f_2) = -x = f_2$$
$$f_1 \cdot f_3 \equiv f_1(f_3) = \frac{1}{x} = f_3 \qquad f_1 \cdot f_4 \equiv f_1(f_4) = -\frac{1}{x} = f_4.$$

Thus, $f_1(x)$ plays the role of an identity. Likewise

$$f_2 \cdot f_1 = -x = f_2 \qquad f_2 \cdot f_2 = -(-x) = x = f_1$$
$$f_2 \cdot f_3 = -\frac{1}{x} = f_4 \qquad f_2 \cdot f_4 = -\left(-\frac{1}{x}\right) = \frac{1}{x} = f_3.$$

Continuing in this manner we find the multiplication table

	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

where $e \equiv f_1$ is the identity. This is a group, since

1. The set is closed under the multiplication operation of substitution.
2. There is an identity (f_1).
3. Each element has a unique inverse (the identity f_1 appears exactly once in each row and column). In fact, each element is its own inverse.
4. The substitution operation is associative.

Since the multiplication table is symmetric about the diagonal, the group is abelian. As shown in Problem 2.19, the matrices

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad d = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

have the same multiplication table under the binary operation of matrix multiplication if we identify $a \leftrightarrow f_1 = e$, $b \leftrightarrow f_2$, $c \leftrightarrow f_3$, and $d \leftrightarrow f_4$.

2.21 This problem is adapted from a discussion in Zee [228]. By matrix multiplication

$$D(u)D(v) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u+v & 1 \end{pmatrix} = D(u+v),$$

which is the multiplication law for the additive group of real numbers. Thus $D(u)$ is a matrix representation of that group. Define a vector with components t and x. Then

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = D(v) \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ vt+x \end{pmatrix},$$

which corresponds to the set of equations

$$t' = t \qquad x' = vt + x$$

that define the Galilean transformations relating time and coordinate for two observers moving along the x-axis with relative velocity v.

2.22 If we define a similarity transform by Eq. (2.12), $D'(x) = S^{-1}D(x)S$, then the product of two transformed matrices is

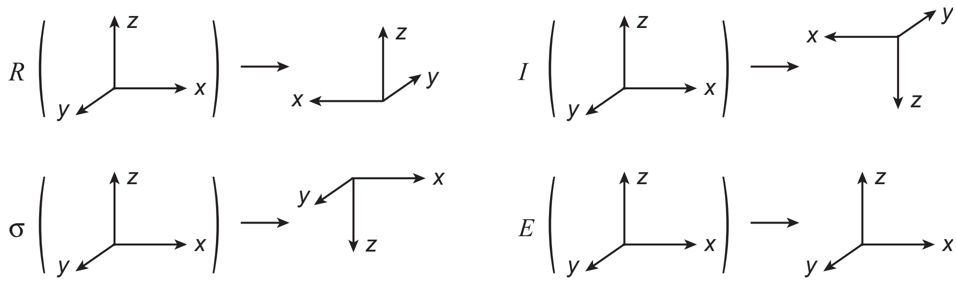
$$\begin{aligned} D'(a) \cdot D'(b) &= S^{-1}D(a)SS^{-1}D(b)S \\ &= S^{-1}D(a) \cdot D(b)S \\ &= S^{-1}D(a \cdot b)S \\ &= D'(a \cdot b), \end{aligned}$$

where in the second line $SS^{-1} = 1$ was used and in the third line Eq. (2.8),

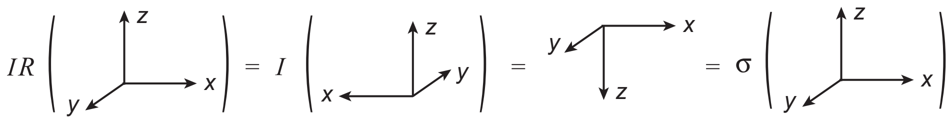
$$D(a) \cdot D(b) = D(a \cdot b),$$

was used. But this means that $D'(x) = S^{-1}D(x)S$ is a representation if $D(x)$ is a representation.

2.23 The actions of the operators R, I, σ, and E on the cartesian axes are



Thus, for example, the product $I \cdot R$ is



so $I \cdot R = \sigma$. Working out the other products we find a multiplication table

C_{2h}	E	R	I	σ
E	E	R	I	σ
R	R	E	σ	I
I	I	σ	E	R
σ	σ	I	R	E

which corresponds to the group labeled C_{2h} in Table 5.1. There are two subgroups with multiplication tables

S_2	E	I
E	E	I
I	I	E

C_2	E	R
E	E	R
R	R	E

Each element of S_2 commutes with each element of C_2 and we can write the C_{2h} operators uniquely as the product of one operation from S_2 and one from C_2,

$$E = E \times E \qquad R = E \times R \qquad I = I \times E \qquad \sigma = I \times R.$$

Therefore, C_{2h} is equivalent to the direct product $C_{2h} = S_2 \times C_2$.

2.24 Under the map $e \to +1$ and $a \to -1$ the C_2 multiplication table given in Box 2.2 becomes

C_2	e	a
e	e	a
a	a	e

$\longrightarrow$

C_2	$+1$	-1
$+1$	$+1$	-1
-1	-1	$+1$

which preserves the group multiplication. Trivially, the map $e \to +1$ and $a \to +1$ satisfies the C_2 multiplication table, so it is a representation also. These representations are both irreducible by application of Eq. (2.31) since the 1×1 matrices have traces ± 1, and from $\sum_\mu n_\mu^2 = n_G$ in Eq. (2.29b) they are the only irreps (up to isomorphisms) because $n_G = 2$ for C_2.

2.25 By explicit matrix multiplication the matrices

$$e = t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a = t_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfy the C_2 multiplication table given in Box 2.2. For example,

$$t_2 \cdot t_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t_1 = e.$$

Therefore, they constitute a representation of the group C_2. To diagonalize the matrix A we must solve the characteristic equation

$$\det(A - \lambda \hat{1}) = 0,$$

where λ is the eigenvalue and $\hat{1}$ is the unit matrix. Inserting the matrix

$$A = t_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

gives two roots for the eigenvalues, $\lambda_\pm = \pm 1$. To find the corresponding eigenvectors we assume a basis

$$\psi = a\psi_+ + b\psi_- \qquad \psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \psi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To determine the coefficients we solve the linear equation

$$A\psi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

From the first row of the matrix equation, choosing $b = 1$,

$$\alpha a + \beta = \lambda a.$$

Choosing the $\lambda = +1$ eigenvalue gives $a_+ = -\beta/(\alpha - 1) = +1$ and choosing the $\lambda = -1$ eigenvalue gives $a_- = -\beta/(\alpha + 1) = -1$, where in the last step we have inserted the specific matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \longrightarrow \quad \alpha = 0 \quad \beta = 1.$$

Thus, up to a normalization the new eigenvectors are

$$\psi_+ = a_+\psi_1 + \psi_2 = \psi_1 + \psi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \psi_- = a_-\psi_1 + \psi_2 = -\psi_1 + \psi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Form a 2×2 matrix S with columns corresponding to these eigenvectors,

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where we've chosen a convenient normalization. Since S is unitary, $S^\dagger = S^{-1}$ and the similarity transformation to the new basis for the matrix t_2 is

$$St_2S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad St_1S^{-1} = t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, in the new diagonalized basis we see that the C_2 representation formed by the 2×2 matrices t_1 and t_2 is reducible, corresponding to a direct sum of the two irreducible representations for C_2 found in Problem 2.24.

2.26 This problem is based on a discussion in Gilmore [72]. Parameterize the complex numbers of unit modulus by

$$C = x + iy = \cos\phi + i\sin\phi.$$

Expand in a power series

$$C = \cos\phi + i\sin\phi = 1 - \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 - \ldots + i\left(\phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 - \ldots\right).$$

Comparing term by term with the exponential power series

$$e^{i\phi} = 1 + i\phi - \frac{1}{2}\phi^2 - \frac{i}{3!}\phi^3 + \frac{1}{4!}\phi^4 - \ldots$$

we see that

$$e^{i\phi} = \cos\phi + i\sin\phi$$

and $e^{i\phi}$ is a faithful mapping for the complex numbers of unit modulus if we restrict $-\pi \leq \phi \leq \pi$. These form a group under multiplication since

1. *Closure:* $e^{i\phi}e^{i\theta} = e^{i(\phi+\theta)} = e^{i\phi'}$ with $\phi' = \phi + \theta$.
2. *Associativity:* $e^{i\phi}(e^{i\theta}e^{i\alpha}) = (e^{i\phi}e^{i\theta})e^{i\alpha} = e^{i(\phi+\theta+\alpha)}$.
3. *Existence of inverse:* $e^{-i\phi}e^{i\phi} = 1$.
4. *Identity:* $e^0e^{i\phi} = e^{i\phi}e^0 = e^{i\phi}$.

Now consider the product

$$(\cos\phi + i\sin\phi)(\cos\theta + i\sin\theta) = e^{i\phi}e^{i\theta} = e^{i(\phi+\theta)}.$$

Multiplying out the left side and rewriting the right side using $e^{i\alpha} = \cos\alpha + i\sin\alpha$ gives

$$\cos\phi\cos\theta - \sin\phi\sin\theta + i(\cos\phi\sin\theta + \sin\phi\cos\theta) = \cos(\phi+\theta) + i\sin(\phi+\theta).$$

Equating separately the real and imaginary parts of the two sides then gives the two trigonometric identies

$$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi+\theta) = \cos\phi\sin\theta + \sin\phi\cos\theta$$

that we sought to prove. This result is a simple example of a general property: all the special functions of mathematical physics are representations of some group and standard identities can be obtained by appropriate operations on representations of the group. See Gilmore [72] for further discussion.

2.27 This problem is adapted from an example in Sternberg [185].

(a) Divide the integers up into four equivalence classes:

$$e \equiv \{0, 4, -4, 8, -8, \ldots\} \qquad a \equiv \{1, 5, -3, 9, -7, \ldots\}$$
$$b \equiv \{2, 6, -2, 10, -6, \ldots\} \qquad c \equiv \{3, 7, -1, 11, -5, \ldots\}$$

and construct a multiplication table of these classes under addition modulo 4. That is, add two integers and subtract an integer multiple of 4 to bring it in the range -4 to $+4$. Consider the product ac. Examples are

$$3 + 1 \bmod 4 = 0 \in e \qquad 5 + 7 \bmod 4 = 0 \in e \qquad 1 + 7 \bmod 4 = 0 \in e.$$

Therefore $ac = ca = e$. Likewise, consider the product ab. For example,

$$1+2 \bmod 4 = 3 \in c \qquad 5+6 \bmod 4 = 3 \in c \qquad -3+6 \bmod 4 = 3 \in c.$$

Therefore, $ab = ba = c$. Carrying out the other possible products gives the multiplication table

Z_4	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

indicating that this is a group and it is abelian.

(b) Consider the matrices

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Matrix multiplication gives results like

$$ab = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = c$$

$$bc = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = a.$$

Carrying out all possibilities, we find that these matrices have the same multiplication table as Z_4. Clearly e plays the role of the identity and the matrix inverse defines the inverse of an element. For example

$$a^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = c,$$

since this leads to

$$aa^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e.$$

Thus, these matrices are a representation of Z_4.

(c) Consider following the rotation operations in a plane

$e =$ rotation by an integer multiple of 2π,
$a =$ rotation counterclockwise by $\frac{\pi}{2}$,
$b =$ rotation counterclockwise by π,
$c =$ rotation counterclockwise by $\frac{3\pi}{2}$.

We can multiply these by implementing first one rotation and then another. For example,

- $ab =$ rotate counterclockwise by $\left(\pi + \frac{\pi}{2}\right) = \frac{3\pi}{2} = c$.
- $bc =$ rotate counterclockwise by $\left(\pi + \frac{3\pi}{2}\right) = \frac{5\pi}{2} = \frac{\pi}{2} = a$.

Carrying out all such multiplication we find the same multiplication table as for Z_4 above. Thus the cyclic group of four-fold rotations in the plane C_4 is homomorphic to Z_4.

(d) The group Z_N can be represented by the elements

$$z_i = e^{2\pi i n/N} \qquad (n = 0, 1, \dots N-1).$$

For Z_4 we have $N = 4$ and $n = 0, 1, 2, 3$, giving the elements

$$e \equiv z_0 = e^0 = 1 \qquad a \equiv z_1 = e^{2\pi i/4} = e^{\frac{\pi}{2}i} \qquad b \equiv z_2 = e^{4\pi i/4} = e^{i\pi} \qquad c \equiv z_3 = e^{\frac{3}{2}\pi i}.$$

Taking products gives results like

$$ac = e^{\left(\frac{\pi}{2}+\frac{3}{2}\pi\right)i} = e^{2\pi i} = e \qquad ab = e^{\left(\frac{\pi}{2}+\pi\right)i} = e^{\frac{3}{2}\pi i} = c.$$

Forming all such products gives once again the multiplication table for Z_4 given above.

2.28 From the constraints $a^2 = b^2 = I = e$ and $ab = ba = c$ we may deduce

$$ab = c \to aab = ac \to ac = b \qquad ba = c \to baa = ca \to ca = b$$
$$ab = c \to abb = cb \to cb = a \qquad ab = c \to cab = c^2 \to b^2 = c^2 = e.$$

The corresponding multiplication table is

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

which is identical to that of the 4-group in Problem 2.9. Thus specifying the constraints $a^2 = b^2 = I = e$ and $ab = ba = c$ is a compact way to specify the content of the multiplication table for the 4-group.

2.29 Let a and b be related by conjugation: $a = gbg^{-1}$, with $a^p = e$ and $b^q = e$. Then

$$e^{1/p} = ge^{1/q}g^{-1} = e^{1/q}.$$

Therefore, $p = q$ for a and b in the same conjugacy class.

2.30 (a) For an arbitrary group element g_a and group identity e,

$$g_a^{-1} e\, g_a = g_a^{-1} g_a = e.$$

Thus e is in a class of its own.

(b) Suppose that for group elements g_a and g_b,

$$g_c^{-1} g_b g_c \equiv g_d \quad \longrightarrow \quad g_b = g_c g_d g_c^{-1}$$
$$g_c^{-1} g_a g_c \equiv g_d \quad \longrightarrow \quad g_a = g_c g_d g_c^{-1}.$$

Multiply $g_d = g_c^{-1} g_b g_c$ from the left by g_c and from the right by g_c^{-1} to give

$$g_c g_d g_c^{-1} = g_c g_c^{-1} g_b g_c g_c^{-1} = g_b.$$

Thus, combining with $g_a = g_c g_d g_c^{-1}$ from above,

$$g_b = g_c g_d g_c^{-1} = g_a.$$

Hence the conjugate classes of g_b and g_a must be identical.

2.31 We may view Z_2 as the group of integers 0 and 1 under addition modulo 2. Trivially this is a group with 0 as the identity and trivially it must be isomorphic to C_2 since all 2-element groups are equivalent. More formally, constructing the multiplication table explicitly gives

Z_2	0	1
0	0	1
1	1	0

Obviously this is isomorphic to the multiplication table in Box 2.2 with the mapping $0 \leftrightarrow e$ and $1 \leftrightarrow a$. The multiplication table for the set $\{1, -1\}$ under ordinary arithmetic multiplication is

Z_2	$+1$	-1
$+1$	$+1$	-1
-1	-1	$+1$

which clearly is isomorphic to C_2 with the mapping $+1 \leftrightarrow e$ and $-1 \leftrightarrow a$. Thus it is isomorphic also to Z_2.

2.32 Suppose a direct product group $G = A \times B$, with elements $g = ab \in G$, where $a \in A$ and $b \in B$, and where the elements of A commute with the elements of B. Consider an arbitrary element a_i of A. Then

$$\begin{aligned} g a_i g^{-1} &= a b a_i (ab)^{-1} \\ &= a b a_i b^{-1} a^{-1} \\ &= a b b^{-1} a_i a^{-1} \\ &= a a_i a^{-1} \in A, \end{aligned}$$

where we've used in line 3 that the elements of A and B commute. Thus A is an invariant subgroup of $G = A \times B$. Likewise, for an arbitrary element b_i of B,

$$\begin{aligned} g b_i g^{-1} &= a b b_i (ab)^{-1} \\ &= a b b_i b^{-1} a^{-1} \\ &= a a^{-1} b b_i b^{-1} \\ &= b b_i b^{-1} \in B, \end{aligned}$$

and B also is an invariant subgroup of $G = A \times B$.

2.33 From the solution of Problem 2.5, $H = \{e, a^2\}$ is an abelian invariant subgroup of $C_4 = \{e, a, a^2, a^3\}$ with the multiplication table for C_4 given in Example 2.15. The independent left cosets of C_4 with respect to H are (see Example 2.16)

$$H = eH = \{e, a^2\} \qquad M \equiv a^3\{e, a^2\} = \{a, a^3\} = aH,$$

giving the coset decomposition

$$C_4 = H + M = \{e, a, a^2, a^3\}.$$

From Eq. (2.27), the coset multiplication law is $pHqH = (pq)H$ and the coset products are (see Example 2.17)

$$H^2 = HH = eHeH = e^2H = H \qquad HM = eHaH = aH = M,$$
$$MH = aHeH = aH = M \qquad M^2 = MM = aHaH = a^2H = H,$$

where we have used $M = aH$ and $a^2H = a^2\{e, a^2\} = \{e, a^2\} = H$. Thus the factor (quotient) group C_4/H has the multiplication table

C_4/H	H	M
H	H	M
M	M	H

which is isomorphic to that for the group C_2.

2.34 The elements M of SL(2,C) are 2×2 matrices of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det M = ad - bc = 1,$$

where the entries a, b, c, and d are arbitrary complex numbers. These matrices form a group under matrix multiplication:

1. Matrix multiplication is associative.
2. Under matrix multiplication the product MM' of two such matrices is a 2×2 matrix and it has unit determinant since $\det(MM') = \det M \det M' = 1$, so closure is satisfied.
3. The unit 2×2 matrix serves as a unique identity.
4. Since $\det M = 1 \neq 0$ the matrices are invertible, so M^{-1} exists and has $\det M^{-1} = 1$:

$$MM^{-1} = 1 \rightarrow \det(MM^{-1}) = 1 \rightarrow \det M \det M^{-1} = 1 \rightarrow \det M^{-1} = 1;$$

   Thus, each $M \in \mathrm{SL}(2,\mathrm{C})$ has a unique inverse $M^{-1} \in \mathrm{SL}(2,\mathrm{C})$, with $MM^{-1} = M^{-1}M$ equal to the unit matrix.

Hence SL(2,C) satisfies the group postulates. In the general case matrices in SL(2,C) may not commute, so the group is non-abelian.

2.35 We follow a discussion in Elliott and Dawber [56]. Define $\psi'(\boldsymbol{r}) = \psi(G_j^{-1}\boldsymbol{r})$. Then from Eq. (2.9),

$$\begin{aligned} T(G_i)T(G_j)\psi(\boldsymbol{r}) &= T(G_i)\psi(G_j^{-1}\boldsymbol{r}) \\ &= T(G_i)\psi'(\boldsymbol{r}) \\ &= \psi'(G_i^{-1}\boldsymbol{r}) \\ &= \psi(G_j^{-1}G_i^{-1}\boldsymbol{r}) \\ &= \psi((G_iG_j)^{-1}\boldsymbol{r}) \\ &= T(G_iG_j)\psi(\boldsymbol{r}). \end{aligned}$$

Thus $T(G_i)T(G_j) = T(G_iG_j)$ and $T(G)$ preserves the group multiplication law (2.8) for G and is a valid representation.