Abstract Algebra (3rd Edition)

Chapter 5.5, Problem 10E

Step-by-step solution

Step 1 of 28

Fundamental theorem of finitely generated abelian groups: [illegible]

Comment

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Lagrange's theorem: [illegible]

Comment

Step 3 of 28

This exercise aims to classify the groups of order 147.

(a) Prove that there are two abelian groups of order 147.

It can be seen that $147 = 3\cdot 7^2$. By Fundamental theorem of finitely generated abelian groups, the abelian groups of order 147 are Z_{147} and $Z_3\times Z_7\times Z_7$. Thus, there are two abelian groups of order 147.

Comment

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(b) Prove that every group of order 147 has a normal Sylow 7-subgroup.

Comment

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Using Sylow's Theorem, the number n_7 of Sylow 7-subgroups divides 3 and is congruent to 1 (mod 7). Hence, $n_7 = 1$. Therefore, the Sylow 7-subgroup of a group of order 147 is unique, thus normal.

Comment

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(c) Prove that there is a unique non-abelian group of order 147 whose Sylow 7-subgroup is cyclic.

Comment

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[illegible]

Comment

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[illegible]

Comment

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[illegible]

Comment

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[illegible]

Comment

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[illegible]

$$\theta(ab^2)=\begin{bmatrix}2&0\\0&4\end{bmatrix}$$
$$\theta(a^2b)=\begin{bmatrix}4&0\\0&2\end{bmatrix}$$
$$\theta(a^2b^2)=\begin{bmatrix}4&0\\0&4\end{bmatrix}$$

Hence, $\ker\theta = 1$, so that $\theta$is injective, and then $|P|=9$. Specifically, $\boxed{P\le GL_2(F_7)}$ is a Sylow 3-subgroup. Note that if $A\le GL_2(F_7)$ is a subgroup of order 3, then *A* is contained in some Sylow 3-subgroup *Q* and thus is conjugate to a subgroup of $\langle\alpha\rangle$.

Comment

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(e) Suppose the group *P* has 4 subgroups of order 3 as follows: $P_1=\langle t_1\rangle, P_2=\langle t_2\rangle, P_3=\langle t_1t_2\rangle$, and $P_4=\langle t_1t_2^2\rangle$. For each $i=1,2,3,4$, let $G_i=(Z_7\times Z_7)ã_{\varphi_i} Z_3$where $\varphi: Z_3\to GL_2(F_7)$is given by $\varphi(x)=\alpha_i$. Here, $\alpha_i = P_i$. For each $i$describe $G_i$in terms of generators and relations and deduce that $G_1\cong G_2$.

Note the following:

Comment

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$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix}$$

Comment

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Since $Z_3=\langle x\rangle$ is cyclic so $\boxed{G_1\cong G_2}$. Each group is generated by the following:

Comment

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$$\mu=\begin{bmatrix}1\\0\end{bmatrix}$$
$$\eta=\begin{bmatrix}0\\1\end{bmatrix}$$
$$\omega = x$$

Comment

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And $\mu\eta=\eta\mu$.

Comment

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For G_1we have the following:

$$\begin{aligned}\omega\mu&=(1,x)\left(\begin{bmatrix}1\\0\end{bmatrix},1\right)\\&=\left(\varphi_1(x)\left(\begin{bmatrix}1\\0\end{bmatrix}\right),x\right)\\&=\left(t_1\begin{bmatrix}1\\0\end{bmatrix},x\right)\\&=\left(\begin{bmatrix}2\\0\end{bmatrix},x\right)\\&=\left(\begin{bmatrix}2\\0\end{bmatrix},1\right)(1,x)\\\omega\mu&=\mu^2\omega\end{aligned}$$

Comment

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And

$$\begin{aligned}\omega\eta&=(1,x)\left(\begin{bmatrix}0\\1\end{bmatrix},1\right)\\&=\left(\varphi_1(x)\left(\begin{bmatrix}0\\1\end{bmatrix}\right),x\right)\\&=\left(t_1\begin{bmatrix}0\\1\end{bmatrix},x\right)\\&=\left(\begin{bmatrix}0\\1\end{bmatrix},x\right)\\&=\left(\begin{bmatrix}0\\1\end{bmatrix},1\right)(1,x)\\\omega\eta&=\eta\omega\end{aligned}$$

Therefore, G_1has the following presentation:

$$\boxed{\langle\mu,\eta,\omega\,|\,\mu^7=\eta^7=\omega^3,\mu\eta=\eta\mu,\omega\eta=\eta\omega,\omega\mu=\mu^2\omega\rangle}$$

For G_3we have the following:

$$\begin{aligned}\omega\mu&=\left(\varphi_3(x)\left(\begin{bmatrix}1\\0\end{bmatrix}\right),x\right)\\&=\left(\begin{bmatrix}2\\0\end{bmatrix},x\right)\\&=\mu^2\omega\end{aligned}$$

And

$$\begin{aligned}\omega\eta&=\left(\varphi_3(x)\left(\begin{bmatrix}0\\1\end{bmatrix}\right),x\right)\\&=\left(\begin{bmatrix}0\\2\end{bmatrix},x\right)\\&=\eta^2\omega\end{aligned}$$

Comment

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Therefore, G_3has the following presentation:

$$\boxed{\langle\mu,\eta,\omega\,|\,\mu^7=\eta^7=\omega^3,\mu\eta=\eta\mu,\omega\eta=\eta^2\omega,\omega\mu=\mu^2\omega\rangle}$$

For G_4in a similar fashion we have $\omega\mu=\mu^2\omega$and $\omega\eta=\eta^4\omega$. Thus G_4has the following presentation: $\boxed{\langle\mu,\eta,\omega\,|\,\mu^7=\eta^7=\omega^3,\mu\eta=\eta\mu,\omega\eta=\eta^4\omega,\omega\mu=\mu^2\omega\rangle}$

Comment

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(f) Prove that $G_1$is not isomorphic to either of $G_3$and G_4.

Let us begin with some lemmas:

Lemma 1: Suppose *G* be a group and let $a,b\in G$such that $ab=b^ka$. Then for all integers $m,n\ge 1$, $a^mb^n=b^{nk^m}a^m$.

Proof: Let us proceed by induction on *m*.

Base case: suppose $m=1$. We proceed by induction on *n*. For the base case, suppose $n=1$. Then:

$$\begin{aligned}a^1b^1&=ab\\&=b^ka\\&=b^{1\cdot k^1}a^1\end{aligned}$$

For the inductive step, let the conclusion holds for some $n\ge 1$. Now,

$$\begin{aligned}a^1b^{n+1}&=ab^nb\\&=b^{nk^1}a^1b\\&=b^{n\cdot k^1}b^ka^1\\&=b^{(n+1)k^1}a^1\end{aligned}$$

Hence, the conclusion holds also for $n+1$. By induction $ab^n=b^{nk}a$ for all *n*.

Comment

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For the inductive step, let the conclusion holds for some $m\ge 1$. We proceed by induction on *n*. For the base case, suppose $n=1$. So,

$$\begin{aligned}a^{m+1}b^1&=aa^mb\\&=ab^{1\cdot k^m}a^m\\&=b^{1\cdot k^m\cdot k}aa^m\\&=b^{1k^{m+1}}a^{m+1}\end{aligned}$$

For the inductive step, let the conclusion holds for some $n\ge 1$. So,

$$\begin{aligned}a^{m+1}b^{n+1}&=aa^mb^nb\\&=ab^{n\cdot k^m}a^mb\\&=b^{n\cdot k^m\cdot k}a^1b^{1\cdot k^m}a^m\\&=b^{nk^{m+1}}b^{k^{m+1}}a^{m+1}\\a^{m+1}b^{n+1}&=b^{(n+1)k^{m+1}}a^{m+1}\end{aligned}$$

By induction we can say that $a^{m+1}b^n=b^{nk^{m+1}}a^{m+1}$ for all *n*, so that the conclusion holds also for $m+1$.Hence, by induction, $a^mb^n=b^{nk^m}a^m$holds for all integers $m,n\ge 1$.

Comment

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Lemma 2: Let *m*, *n* are positive integers such that $m+a\cdot 2^n\equiv a+m\cdot 2^b\pmod 7$for all positive integers *a* and *b*. Then, $m\equiv 0\pmod 7$and $n\equiv 0\pmod 3$.

Proof: Suppose $b=3$and $0<a<7$. Then, $m+a\cdot 2^n\equiv a+m\pmod 7$, so that $a\cdot 2^n\equiv a\pmod 7$. Now *a* is invertible mod 7, so that $2^n\equiv 1\pmod 7$. Since $|2|=3$in $Z/(7)$, we have $n\equiv 0\pmod 3$. Now we have $m+a\equiv a+m\cdot 2^b\pmod 7$, so that $m\equiv m\cdot 2^b\pmod 7$. If $m\not\equiv 0\bmod 7$, then $1\equiv 2^b\pmod 7$for all integers *b*, which is not correct. Hence, $m\equiv 0\bmod 7$.

Now, we will calculate the centers of $G_1, G_3$and G_4.

Note from the presentation of G_1that every element in this group can be written in the form $\mu^i\eta^k\omega^k$for some $0<i,j\le 7$and $0<k\le 3$. There are 147 such forms, and because it is known that $|G_1|=147$, then in fact each element can be written in this form uniquely.

Consider that $x=\mu^a\eta^b\omega^c\in Z(G_1)$. Then for every element $\mu^i\eta^k\omega^k$and by using Lemma 1 we have the following:

$$\begin{aligned}\mu^a\eta^b\omega^c\mu^i\eta^j\omega^k&=\mu^{a+i2^c}\eta^{b+j}\omega^{c+k}\\&=\mu^i\eta^j\omega^k\mu^a\eta^b\omega^c\\&=\mu^{a+i2^k}\eta^{j+b}\omega^{k+c}\end{aligned}$$

Hence, we have that, for some fixed positive integers *a* and *c*, $a+i2^c\equiv i+a2^k\pmod 7$for all positive integers *i* and *k*. By Lemma 2, we have $a\equiv 0\bmod 7$and $c\equiv 0\bmod 3$. Hence $x\in\langle\eta\rangle$. Also, $\langle\eta\rangle\le Z(G_1)$, so $Z(G_1)=\langle\eta\rangle$.

Comment

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Again, it can be seen that every element of G_4can be written in the form $\mu^i\eta^k\omega^k$for some $0<i,j\le 7$and $0<k\le 3$, and it is known that $|G_3|=147$, every element can be written uniquely in this form. Consider that $x=\mu^a\eta^b\omega^c\in Z(G_3)$. Then for every element $\mu^i\eta^k\omega^k$,following is equivalent given as follows:

$$\begin{aligned}\mu^a\eta^b\omega^c\mu^i\eta^j\omega^k&=\mu^{a+i2^c}\eta^{b+j2^c}\omega^{c+k}\\&=\mu^i\eta^j\omega^k\mu^a\eta^b\omega^c\\&=\mu^{i+a2^k}\eta^{j+b2^k}\omega^{k+c}\end{aligned}$$

Comparing exponents and using Lemma 2, we have $a\equiv b\equiv 0\bmod 7$ and $c\equiv 0\bmod 3$. Hence $x=1$and we have $Z(G_3)=1$.

Comment

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Again, we can see that every element of G_4can be written in the form $\mu^i\eta^k\omega^k$for some $0<i,j\le 7$and $0<k\le 3$, and it is known that $|G_4|=147$, every element can be written uniquely in this form. Consider $x=\mu^a\eta^b\omega^c\in Z(G_4)$. Then for every element $\mu^i\eta^k\omega^k$, following is equivalent given as follows:

$$\begin{aligned}\mu^a\eta^b\omega^c\mu^i\eta^j\omega^k&=\mu^{a+i2^c}\eta^{b+j4^c}\omega^{c+k}\\&=\mu^i\eta^j\omega^k\mu^a\eta^b\omega^c\\&=\mu^{i+a2^k}\eta^{j+b4^k}\omega^{k+c}\end{aligned}$$

Comparing the exponents of $\mu$and by Lemma 2 we have $a\equiv 0\bmod 7$and $c\equiv 0\bmod 3$. Comparing the exponents of η, we have $b+j\equiv j+b4^k\pmod 7$, so that $b\equiv b4^k\pmod 7$. If $b\not\equiv 0\bmod 7$, then $1\equiv 4^k\bmod 7$for all $0<k\le 3$, which is not correct. Hence, $b\equiv 0\bmod 7$. Thus, $x=1$and we have $Z(G_4)=1$.

Specifically, it is seen that $\boxed{G_1\not\cong G_3}$and $\boxed{G_1\not\cong G_4}$since $Z(G_3)$and $Z(G_4)$are trivial while $Z(G_1)$is not.

Comment

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(g) Prove that $G_3$is not isomorphic to G_4.

It is known that the Sylow 7-subgroups of $G_3$and $G_4$are unique, so that every subgroup of order 7 in each is contained in $H=Z_7^2$. Recall that there are 8 such subgroups given as follows: $\langle\mu\rangle,\langle\mu\eta\rangle,\langle\mu\eta^2\rangle,\langle\mu\eta^3\rangle,\langle\mu\eta^4\rangle,\langle\mu\eta^5\rangle,\langle\mu\eta^6\rangle$and $\langle\eta\rangle$. Note the following:

$$\omega\eta\omega^{-1}=\eta^2\in\langle\eta\rangle$$

And

$$
\begin{aligned}
&= (\mu\eta^{t}) \\
&\in \langle \mu\eta^{t} \rangle
\end{aligned}
$$

Hence, every subgroup of G_3 order 7 is normal there.

Comment

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On the other hand, in G_4, we have

$$
\begin{aligned}
\omega\mu\eta\omega^{-1} &= \mu^2\omega\eta\omega^{-1} \\
&= \mu^2\eta^4 \\
&= (\mu\eta^2)^2 \\
&\in \langle \mu\eta^2 \rangle
\end{aligned}
$$

By Lagrange $\langle \mu\eta \rangle \cap \langle \mu\eta^2 \rangle = 1, \omega\mu\eta w^{-1} \notin \langle \mu\eta \rangle$. Hence, $\langle \mu\eta \rangle$ is an order 7 subgroup which is not normal in G_4. Thus $\boxed{G_3 \not\cong G_4}$.

Comment

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(h) Classify the groups of order 147 by showing that the six non-isomorphic groups described above are all the groups of order 147.

Suppose *G* be a group of order $147 = 3 \cdot 7^2$. If *G* is abelian, then *G* is isomorphic to one of Z_{147} and $Z_{21} \times Z_7$. If *G* is non-abelian, then by part (b), *G* has a normal Sylow 7-subgroup *H*. Suppose $K \cong Z_3$ be any Sylow 3-subgroup of *H*. By Lagrange, $H \cap K = 1$, so that $G = HK$. With the help of the recognition theorem for semi-direct products, $G_i \cong H \tilde{a}_{\varphi} K$, where $K = Z_3$, $|H| = 49$, and $\varphi : K \to \text{Aut}(H)$.

Comment

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Hence, classifying the non-abelian groups of order 147 is equivalent to determining the non-isomorphic groups constructed in this manner. By part (c), there is an essentially unique non-abelian group of the form $Z_{49} \tilde{a}_{\varphi} Z_3$. Consider that $H = Z_7^2$. Since *G* is non-abelian, φ is nontrivial. As *K* is simple, we have $\ker\varphi = 1$; thus $\text{im}\varphi$ is a subgroup of order 3 in $\text{Aut}(H) \cong GL_2(\mathrm{F}_7)$. By part (d), $\text{im}\varphi$ is conjugate to one of the four subgroups of the Sylow 3-subgroup $P \le GL_2(\mathrm{F}_7)$ identified above. Since *K* is cyclic, $H \tilde{a}_{\varphi} K$ is isomorphic to one of the groups G_i identified above. It is showed in parts (e), (f), and (g) that three of these are distinct. Thus the groups of order 147 up to isomorphism are as follows.

In all case, let $Z_3 = \langle x \rangle, Z_{49} = \langle y \rangle$ and $Z_7^2 = \langle a \rangle \times \langle b \rangle$.

1. $\boxed{Z_{147}}$
2. $\boxed{Z_{21} \times Z_7}$
3. $\boxed{Z_{49} \tilde{a}_{\varphi} Z_3}$ where $\psi(x)(y) = y^{-4}$
4. $\boxed{Z_7^2 \tilde{a}_{\varphi_1} Z_3}$ where $\varphi_1(x)(a) = a^2 \text{and } \varphi_1(x)(b) = b$
5. $\boxed{Z_7^2 \tilde{a}_{\varphi_2} Z_3}$ where $\varphi_2(x)(a) = a^2 \text{and } \varphi_2(x)(b) = b^2$
6. $\boxed{Z_7^2 \tilde{a}_{\varphi_3} Z_3}$ where $\varphi_3(x)(a) = a^2 \text{and } \varphi_3(x)(b) = b^4$

Comment