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Abstract Algebra (3rd Edition)

Chapter 2.5, Problem 6E

Step-by-step solution

Step 1 of 7

By using the given lattices we have to find the centralizers of every element in the following groups:

D_8

Q_8

S_3

D_{16}

Comment

Step 2 of 7

Lemma 1: Let us consider that, G is a group and that $x \in G$ and $z \in Z(G)$.

Thus, we have

$C_G(xz) = C_G(x)$

Now we have to prove Lemma 1:

Proof: If we consider that $y \in C_G(x)$ then we have $yxz = xyz = xzy$, such that $y \in C_G(xz)$.

Now let us suppose that $y \in C_G(xz)$. Then we have $yxz = xzy = xyz$, such that by cancellation we get, $yx = xy$ and thus we have $y \in C_G(x)$.

Comment

Step 3 of 7

Lemma 2: Let G be a group and $x, g \in G$. Then

$C_G(gxg^{-1}) = g(C_G(x))g^{-1}$

Now we have to prove the Lemma 2.

Proof: If we consider that $y \in C_G(x)$ then we have $y = gzg^{-1}$ such that $z \in C_G(x)$.

Thus we have,

$$\begin{aligned}(gzg^{-1})(gxg^{-1}) &= gzxg^{-1} \\ &= gxzg^{-1} \\ &= (gxg^{-1})(gzg^{-1})\end{aligned}$$

such that $y \in C_G(gxg^{-1})$.

Consider that $y \in C_G(gxg^{-1})$. Thus, we have

$ygxg^{-1} = gxg^{-1}y$

Hence,

$(g^{-1}yg)x = x(g^{-1}yg)$

Hence we get $g^{-1}yg \in C_G(x)$, and we have $y \in gC_G(x)g^{-1}$.

Comment

Step 4 of 7

Lemma 3: Let G be a group and let $g \in G, A \subseteq G$. Then

$g\langle A\rangle g^{-1} = \langle gAg^{-1}\rangle$

Now we have to prove the Lemma 3:

Proof: If we consider that $y \in g\langle A\rangle g^{-1}$ then we have $y = gzg^{-1}$ where $z \in \langle A\rangle$. We know that $z = a_1a_2 \dots a_n$ where for each i, either a_i or a_i^{-1} is in A.

Thus, we have $z = a_1g^{-1}ga_2g^{-1}g \dots g^{-1}ga_n$, such that

$gzg^{-1} = (ga_1g^{-1})(ga_2g^{-1}\dots(ga_ng^{-1})$

Therefore, we have

$y \in \langle gAg^{-1}\rangle$

Comment

Step 5 of 7

Lemma 4: Let G be a group and let $a, b \in G$. If $\langle a\rangle = \langle b\rangle$, then

$C_G(a) = C_G(b)$

Now we have to prove Lemma 4:

Proof: we know that $b = a^k$ for some k, such that if $xa = ax$, then we have

$xb = bx$

Therefore,

$C_G(a) \le C_G(b)$

Comment

Step 6 of 7

Now consider $G = D_8$. We know that,

$Z(D_8) = \{1, r^2\}$

x	Reasoning	$C_G(x)$
1	$1 \in Z(D_8)$	D_8
r	$\langle r\rangle \le C_G(r)$, so $C_G(r)$ is either $\langle r\rangle$ or D_8. But $sr \ne rs$, so $s \notin C_G(r)$, hence $C_G(r) \ne D_8$.	$\langle r\rangle$
r^2	$r^2 \in Z(D_8)$	D_8
r^3	$r^3 = r^{-1}$	$\langle r\rangle$
s	$\langle s\rangle \le C_G(s)$ and $\langle r^2\rangle \le C_G(s)$ since $r^2 \in Z(G)$, so $C_G(s)$ is either $\langle s, r^2\rangle$ or D_8. But $r \notin C_G(s)$ since $sr \ne rs$.	$\langle s, r^2\rangle$
sr	$\langle sr\rangle \le C_G(sr)$ and $\langle r^2\rangle \le C_G(sr)$ since $r^2 \in Z(G)$, so $C_G(sr)$ is either $\langle sr, r^2\rangle$ or D_8. But $rsr = s \ne sr^2$, so $r \notin C_G(sr)$.	$\langle sr, r^2\rangle$
sr^2	$sr^2 = s \cdot r^2$, so $C_G(sr^2) = C_G(s)$ by Lemma 1	$\langle s, r^2\rangle$
sr^3	$sr^3 = sr \cdot r^2$, so $C_G(sr^3) = C_G(sr)$ by Lemma	$\langle sr, r^2\rangle$

$G = Q_8$. We know that,

$Z(Q_8) = \{1, -1\}$

x	Reasoning	$C_G(x)$
1	$1 \in Z(G)$	Q_8
-1	$-1 \in Z(G)$	Q_8
i	$\langle i\rangle \le C_G(i)$ so $C_G(i)$ is either $\langle i\rangle$ or Q_8. But $j \notin C_G(i)$ since $ij = k \ne -k = ji$.	$\langle i\rangle$
$-i$	$-i = i^{-1}$, so $C_G(-i) = C_G(i)$	$\langle i\rangle$
j	$\langle j\rangle \le C_G(j)$ so $C_G(j)$ is either $\langle j\rangle$ or Q_8. But $k \notin C_G(j)$ since $jk = i \ne -i = kj$.	$\langle j\rangle$
$-j$	$-j = j^{-1}$, so $C_G(-j) = C_G(j)$.	$\langle j\rangle$
k	$\langle k\rangle \le C_G(k)$, so $C_G(k)$ is either $\langle k\rangle$ or Q_8. But $i \notin C_G(k)$ since $ki = j \ne -j = ik$.	$\langle k\rangle$
$-k$	$-k = k^{-1}$, so $C_G(-k) = C_G(k)$	$\langle k\rangle$

Now consider S_3:

x	Reasoning	$C_G(x)$
1		S_6
$(1\,2)$	$\langle(1\,2)\rangle \le C_G((1\,2))$, so $C_G((1\,2))$ is either S_6 or $\langle(1\,2)\rangle$. Thus we have S_6 which does not commute with $(1\,2)\,G$.	$\langle(1\,2)\rangle$
$(1\,3)$	We know that $(1\,3) = (2\,3)(1\,2)(2\,3)$, so we can apply Lemmas 2 and 3.	$\langle(1\,3)\rangle$
$(2\,3)$	We have $(2\,3) = (1\,3)(1\,2)(1\,3)$, so we can apply Lemmas 2 and 3.	$\langle(2\,3)\rangle$
$(1\,2\,3)$	$\langle(1\,2\,3) \le C_G((1\,2\,3))$, so $C_G((1\,2\,3))$ is either $\langle(1\,2\,3)\rangle$ or G. But $(1\,2)$ does not commute with $(1\,2\,3)$.	$\langle(1\,2\,3)\rangle$
$(1\,3\,2)$	$(1\,3\,2) = (1\,2\,3)^{-1}$	$\langle(1\,2\,3)\rangle$

Comment

Step 7 of 7

Now consider D_{16}:

x		$C_G(x)$
1		D_{16}
r	$\langle r\rangle \le D_{16}$, so $C_G(r)$ is either D_{16} or $\langle r\rangle$. But $rs \ne sr$.	$\langle r\rangle$
r^2	$\langle r^2\rangle \le C_G(r^2)$, so $C_G(r)$ is either $\langle r^2\rangle, \langle s, r^2\rangle, \langle r\rangle, \langle sr, r^2\rangle$, or D_{16}. We know that $sr^2 \ne r^2s$, and $rr^2 = r^2r$, and $srr^2 \ne r^2sr$.	$\langle r\rangle$
r^3	$\langle r^3\rangle = \langle r\rangle$	$\langle r\rangle$
r^4	$r^4 \in Z(G)$	D_{16}
r^5	$\langle r^5\rangle = \langle r\rangle$	$\langle r\rangle$
r^6	$r^6 = r^2r^4$	$\langle r\rangle$
r^7	$r^7 = r^{-1}$.	$\langle r\rangle$
s	$\langle s\rangle \le C_G(s)$, so $C_G(s)$ is either $\langle s\rangle, \langle s, r^4\rangle, \langle s, r^2\rangle$, or D_{16}. Now $\langle r^4\rangle \le C_G(s)$, and $r^2s \ne sr^2$.	$\langle s, r^4\rangle$
sr	$\langle sr\rangle \le C_G(sr)$, so $C_G(sr)$ is either $\langle sr\rangle, \langle sr, r^4\rangle, \langle sr, r^2\rangle$, or D_{16}. We know that $r^4sr = srr^4$ and $srr^2 \ne r^2sr$.	$\langle sr, r^4\rangle$
sr^2	$\langle sr^2\rangle \le C_G(sr^2)$, so $C_G(sr^2)$ is either $\langle sr^2\rangle, \langle sr^2, r^4\rangle, \langle s, r^2\rangle$, or D_{16}. We know that $r^4sr^2 = sr^2r^4$ and $r^2sr^2 \ne sr^2r^2$.	$\langle sr^2, r^4\rangle$
sr^3	$\langle sr^3\rangle \le C_G(sr^3)$ and $\langle r^4\rangle \le C_G(sr^3)$. Hence we have $r^2sr^3 \ne sr^3r^2$.	$\langle sr^3, r^4\rangle$
sr^4	Lemma 1	$\langle s, r^4\rangle$
sr^5	Lemma 1	$\langle sr, r^4\rangle$
sr^6	Lemma 1	$\langle sr^2, r^4\rangle$
sr^7	Lemma 1	$\langle sr^3, r^4\rangle$

Comment