Abstract Algebra (3rd Edition)

Chapter 7.6, Problem 8E

Step-by-step solution

Step 1 of 16

Consider *I* be a nonempty partially ordered set, and Suppose $\{A_i\}_I$ be a collection of abelian groups. Let us suppose that *I* is directed; For all $i, j \in I$, there exists $k \in I$ with $i, j \le k$.

Consider that for every pair of indices $i, j \in I$ with $i \le j$, there is a map $\rho_{i,j} : A_i \longrightarrow A_j$ such that the following hold: (1) $\rho_{j,k} \circ \rho_{i,j} = \rho_{i,k}$ whenever $i \le j \le k$ and (2) $\rho_{i,i} = 1$ for all $i \in I$.

Suppose $B = \bigcup_I A_i \times \{i\}$ be the disjoint union of the A_i. Define a relation σ on *B* as follows:

Comment

Step 2 of 16

$(a,i)\sigma(b,j)$ if and only if there exists $k \in I$ such that $i, j \le k$ and $\rho_{i,k}(a) = \rho_{j,k}(b)$.

Comment

Step 3 of 16

(a) Show that σ is an equivalence relation on *B*. We define $\varinjlim A_i = B/\sigma$.

To show that σ is equivalence, we need to verify that it is reflexive, symmetric, and transitive.

1.(σ is reflexive) Consider $(a,i) \in B$. Note that $i \le i$, and that

$$\begin{aligned}\rho_{i,i}(a) &= a \\ &= \rho_{i,i}(a)\end{aligned}$$

Thus $(a,i)\sigma(a,i)$, and hence σ is reflexive.

2.(σ is symmetric) Consider $(a,i)\sigma(b,j)$. Then there exists $k \ge i, j$ such that $\rho_{i,k}(a) = \rho_{j,k}(b)$. Certainly $\rho_{j,k}(b) = \rho_{i,k}(a)$, so that $(b,j)\sigma(a,i)$. Hence σ is symmetric.

3.(σ is transitive) Suppose $(a,i)\sigma(b,j)$ and $(b,j)\sigma(c,k)$. Then there exist $l \ge i, j$ such that $\rho_{i,l}(a) = \rho_{j,l}(b)$ and $m \ge j,k$ such that $\rho_{j,m}(b) = \rho_{k,m}(c)$. Since *I* is a directed poset, there exists $t \in I$ such that $t \ge l, m$. Now

$$\begin{aligned}\rho_{i,t}(a) &= \rho_{l,t}(\rho_{i,l}(a)) \\ &= \rho_{l,t}(\rho_{j,l}(b)) \\ &= \rho_{j,t}(b) \\ &= \rho_{m,t}(\rho_{j,m}(b))\end{aligned}$$

$$\begin{aligned}\rho_{i,t}(a) &= \rho_{m,t}(\rho_{k,m}(c)) \\ &= \rho_{k,t}(c)\end{aligned}$$

Thus $(a,i)\sigma(c,k)$, and hence σ is transitive. Hence σ is an equivalence relation.

Comment

Step 4 of 16

(b) Consider $[x]_\sigma$ denote the class of *x* in $\varinjlim A_i$ and define $\rho_i : A_i \to \varinjlim A_i$ by $\rho_i(a) = [(a,i)]_\sigma$. Show that if each $\rho_{i,j}$ is injective, then ρ_i is also injective for all *i*.

Suppose that the $\rho_{i,j}$ are all injective. Choose $i \in I$, and Consider $a, b \in A_i$ such that $\rho_i(a) = \rho_i(b)$.Then $[(a,i)]_\rho = [(b,i)]_\rho$.For some $k \ge i$,, we have $\rho_{i,k}(a) = \rho_{i,k}(b)$. Since $\rho_{i,k}$ is injective, $a = b$. Hence ρ_i is injective.

Comment

Step 5 of 16

(c) Assume that the $\rho_{i,j}$ are all group homomorphism. For $[(a,i)]_\sigma, [(b,j)]_\sigma \in \varinjlim A_i$, show that the operation $[(a,i)]_\sigma + [(b,j)]_\sigma = [(\rho_{i,k}(a) + \rho_{j,k}(b), k)]_\sigma$, where *k* is any upper bound of *i* and *j*, is well defined and makes $\varinjlim A_i$ an abelian group. Deduce that the ρ_i are group homomorphism.

Consider that the $\rho_{i,j}$ are all group homomorphism. First we show that $+$ is well defined.

Suppose $[(a_1,i_1)]_\sigma = [(a_2,i_2)]_\sigma$ and $[(b_1,j_1)]_\sigma = [(b_2,j_2)]_\sigma$. Then there exists $s \ge i_1, i_2$ such that $\rho_{i_1,s}(a_1) = \rho_{i_2,s}(a_2)$ and $r \ge j_1, j_2$ such that $\rho_{j_1,r}(b_1) = \rho_{j_2,r}(b_2)$. Now choose arbitrary $k_1 \ge i_1, j_1$ and $k_2 \ge i_2, j_2$. Again choose $t \ge k_1, k_2, r, s$. See the following:

$$\begin{aligned}\rho_{k_1,t}(\rho_{i_1,k_1}(a_1) + \rho_{j_1,k_1}(b_1)) &= \rho_{k_1,t}(\rho_{i_1,k_1}(a_1)) + \rho_{k_1,t}(\rho_{j_1,k_1}(b_1)) \\ &= \rho_{i_1,t}(a_1) + \rho_{j_1,t}(b_1) \\ &= \rho_{s,t}(\rho_{i_1,s}(a_1)) + \rho_{r,t}(\rho_{j_1,r}(b_1)) \\ &= \rho_{s,t}(\rho_{i_2,s}(a_2)) + \rho_{r,t}(\rho_{j_2,r}(b_2))\end{aligned}$$

$$\begin{aligned}\rho_{k_1,t}(\rho_{i_1,k_1}(a_1) + \rho_{j_1,k_1}(b_1)) &= \rho_{i_2,t}(a_2) + \rho_{j_2,t}(b_2) \\ &= \rho_{k_2,t}(\rho_{i_2,k_2}(a_2)) + \rho_{k_2,t}(\rho_{j_2,k_2}(b_2)) \\ &= \rho_{k_2,t}(\rho_{i_2,k_2}(a_2) + \rho_{j_2,k_2}(b_2))\end{aligned}$$

Hence, $(\rho_{i_1,k_1}(a_1) + \rho_{j_1,k_1}(b_1), k_1)\sigma(\rho_{i_2,k_2}(a_2) + \rho_{j_2,k_2}(b_2), k_2)$, and

$$\begin{aligned}[(a_1,i_1)]_\sigma + [(b_1,j_1)]_\sigma &= [(a_2,i_2)]_\sigma \\ &= [(b_2,j_2)]_\sigma\end{aligned}$$

Thus $+$ is well-defined.

Next we show that $\left(\varinjlim A_i, +\right)$ is an abelian group.

(1) ($+$ is associative) Consider $[(a,i)]$, $[(b,j)]$, and $[(c,k)]$ be in $\varinjlim A_i$, and Suppose $l \ge i, j$, $t \ge j, k$, and $m \ge l, t$. Then we have the following.

$$\begin{aligned}[(a,i)] + [(b,j)] + [(c,k)] &= [\rho_{i,l}(a) + \rho_{j,l}(b)] + [(c,k)] \\ &= [(\rho_{l,m}(\rho_{i,l}(a) + \rho_{j,l}(b)) + \rho_{k,m}(c), m)] \\ &= [(\rho_{l,m}(\rho_{i,l}(a)) + \rho_{l,m}(\rho_{j,l}(b)) + \rho_{k,m}(c), m)] \\ &= [(\rho_{i,m}(a) + \rho_{j,m}(b) + \rho_{k,m}(c), m)]\end{aligned}$$

$$\begin{aligned}[(a,i)] + [(b,j)] + [(c,k)] &= [(\rho_{i,m}(a) + \rho_{t,m}(\rho_{j,t}(b)) + \rho_{t,m}(\rho_{j,t}(c)), m)] \\ &= [(\rho_{i,m}(a) + \rho_{t,m}(\rho_{j,t}(b) + \rho_{k,t}(c)), m)] \\ &= [(a,i)] + [(\rho_{j,t}(b) + \rho_{k,t}(c), t)] \\ &= [(a,i)] + ([(b,j)] + [(c,k)])\end{aligned}$$

So $+$ is associative.

Comment

Step 6 of 16

(2) For all $i, j \in I$, there exists $k \ge i, j$, and $\rho_{i,k}(0) = \rho_{j,k}(0)$ since the $\rho_{i,j}$ are group homomorphism. Hence $[(0,i)] = [(0,j)]$ for all i, j. Consider $0 = [(0,i)]$. Suppose $[(a,i)] \in \varinjlim A_i$. Then

$$\begin{aligned}0 + [(a,i)] &= [(0,i)] + [(a,i)] \\ &= [(\rho_{i,i}(0) + \rho_{i,i}(0), i)] \\ &= [(0 + a, i)] \\ &= [(a,i)]\end{aligned}$$

Thus, $[(a,i)] + 0 = [(a,i)]$. Hence $0 = [(0,i)]$ is an additive identity element.

Comment

Step 7 of 16

3. Consider $[(a,i)] \in \varinjlim A_i$. Note that

$$\begin{aligned}[(a,i)] + [(-a,i)] &= [(\rho_{i,i}(a) + \rho_{i,i}(-a), i)] \\ &= [(a - a, i)] \\ &= [(0,i)] \\ &= 0\end{aligned}$$

Hence every element of $\varinjlim A_i$ has an additive inverse.

Comment

Step 8 of 16

4. Suppose $[(a,i)], [(b,j)] \in \varinjlim A_i$, and Consider $k \ge i, j$. Then

$$\begin{aligned}[(a,i)] + [(b,j)] &= [(\rho_{i,k}(a) + \rho_{j,k}(b), k)] \\ &= [(\rho_{j,k}(b) + \rho_{i,k}(a), k)] \\ &= [(b,j)] + [(a,i)]\end{aligned}$$

Hence $+$ is commutative.

Thus $\left(\varinjlim A_i, +\right)$ is an abelian group. Finally, we show that each $\rho_i : A_i \to \varinjlim A_i$ is a group homomorphism. Suppose $a, b \in A_i$. Then

$$\begin{aligned}\rho_i(a+b) &= [(a+b, i)] \\ &= [(\rho_{i,i}(a) + \rho_{i,i}(b), i)] \\ &= [(a,i)] + [(b,i)] \\ &= \rho_i(a) + \rho_i(b)\end{aligned}$$

Hence ρ_i is a group homomorphism for all *i*.

Comment

Step 9 of 16

(d) Prove that if all the A_i are commutative rings with $1 \ne 0$ and all the $\rho_{i,j}$ are unital ring homomorphism, then $\varinjlim A_i$ may likewise be given the structure of a commutative ring with $1 \ne 0$ such that the ρ_i are all ring homomorphism.

Define an operator on $\varinjlim A_i$ as follows: $[(a,i)] \cdot [(b,j)] = [(\rho_{i,k}(a) \cdot \rho_{j,k}(b), k)]$, where *k* is any upper bound of *i* and *j* in *I*. See this.

(1) ($(\cdot)$ is well defined) Consider $[(a_1,i_1)] = [(a_2,i_2)]$ and $[(b_1,j_1)] = [(b_2,j_2)]$. Then there exist $r \ge i_1, i_2$ and $s \ge j_1, j_2$ such that $\rho_{i_1,r}(a_1) = \rho_{i_2,r}(a_2)$ and $\rho_{j_1,s}(b_1) = \rho_{j_2,s}(b_2)$. Choose $k_1 \ge i_1, j_1$, $k_2 \ge i_2, j_2$, and $t \ge k_1, k_2$. Now,

$$\begin{aligned}\rho_{k_1,t}(\rho_{i_1,k_1}(a_1) \cdot \rho_{j_1,k_1}(b_1)) &= \rho_{k_1,t}(\rho_{i_1,k_1}(a_1)) \cdot \rho_{k_1,t}(\rho_{j_1,k_1}(b_1)) \\ &= \rho_{i_1,t}(a_1) \cdot \rho_{j_1,t}(b_1) \\ &= \rho_{r,t}(\rho_{i_1,r}(a_1)) \cdot \rho_{s,t}(\rho_{j_1,s}(b_1)) \\ &= \rho_{r,t}(\rho_{i_2,r}(a_2)) \cdot \rho_{s,t}(\rho_{j_2,s}(b_2))\end{aligned}$$

$$\begin{aligned}\rho_{k_1,t}(\rho_{i_1,k_1}(a_1) \cdot \rho_{j_1,k_1}(b_1)) &= \rho_{i_2,t}(a_2) \cdot \rho_{j_2,t}(b_2) \\ &= \rho_{k_2,t}(\rho_{i_2,k_2}(a_2)) \cdot \rho_{k_2,t}(\rho_{j_2,k_2}(b_2)) \\ &= \rho_{k_2,t}(\rho_{i_2,k_2}(a_2) \cdot \rho_{j_2,k_2}(b_2))\end{aligned}$$

Thus $(\rho_{i_1,k_1}(a_1) \cdot \rho_{j_1,k_1}(b_1), t)\sigma(\rho_{i_2,k_2}(a_2) \cdot \rho_{j_2,k_2}(b_2), t)$, and particularly, $[(a_1,i_1)] \cdot [(b_1,j_1)] = [(a_2,i_2)] \cdot [(b_2,j_2)]$. So $(\cdot)$ is well-defined.

Comment

Step 10 of 16

(2) ($(\cdot)$ is associative) Suppose $[(a,i)], [(b,j)], [(c,k)] \in \varinjlim A_i$. Consider $r \ge i, j$, $s \ge j, k$, and $t \ge r, s$. Then we have the following:

$$\begin{aligned}[(a,i)] \cdot [(b,j)] \cdot [(c,k)] &= [\rho_{i,r}(a) \cdot \rho_{j,r}(b), r] \cdot [(c,k)] \\ &= [(\rho_{r,t}(\rho_{i,r}(a) \cdot \rho_{j,r}(b)) \cdot \rho_{k,t}(c), t)] \\ &= [(\rho_{r,t}(\rho_{i,r}(a)) \cdot \rho_{r,t}(\rho_{j,r}(b)) \cdot \rho_{k,t}(c), t)] \\ &= [(\rho_{i,t}(a) \cdot \rho_{j,t}(b) \cdot \rho_{k,t}(c), t)]\end{aligned}$$

$$\begin{aligned}[(a,i)] \cdot [(b,j)] \cdot [(c,k)] &= [(\rho_{i,t}(a) \cdot \rho_{s,t}(\rho_{j,s}(b)) \cdot \rho_{s,t}(\rho_{k,s}(c)), t)] \\ &= [(\rho_{i,t}(a) \cdot \rho_{s,t}(\rho_{j,s}(b) \cdot \rho_{k,s}(c)), t)] \\ &= [(a,i)] \cdot [(\rho_{j,s}(b) \cdot \rho_{k,s}(c), s)] \\ &= [(a,i)] \cdot ([(b,j)] \cdot [(c,k)])\end{aligned}$$

Hence $(\cdot)$ is associative.

Comment

Step 11 of 16

(3) ($(\cdot)$ distributes over $(+)$) We will show that $(\cdot)$ distributes over $(+)$ on the left; distributes on the right is similar. Let $[(a,i)], [(b,j)], [(c,k)] \in \varinjlim A_i$, Suppose $r \ge j, k$, and let $t \ge i, r$. Then we have the following:

$$\begin{aligned}[(a,i)] \cdot ([(b,j)] + [(c,k)]) &= [(a,i)] \cdot [(\rho_{j,r}(b) + \rho_{k,r}(c), r)] \\ &= [(\rho_{i,t}(a) \cdot \rho_{r,t}(\rho_{j,r}(b) + \rho_{k,r}(c)), t)] \\ &= [(\rho_{i,t}(a) \cdot (\rho_{r,t}(\rho_{j,r}(b)) + \rho_{r,t}(\rho_{k,r}(c))), t)] \\ &= [(\rho_{i,t}(a) \cdot (\rho_{j,t}(b) + \rho_{k,t}(c)), t)]\end{aligned}$$

$$\begin{aligned}[(a,i)] \cdot ([(b,j)] + [(c,k)]) &= [(\rho_{i,t}(a) \cdot \rho_{j,t}(b) + \rho_{i,t}(a) \cdot \rho_{k,t}(c), t)] \\ &= [(\rho_{t,t}(\rho_{i,t}(a) \cdot \rho_{j,t}(b)) + \rho_{t,t}(\rho_{i,t}(a) \cdot \rho_{k,t}(c)), t)] \\ &= [(\rho_{i,t}(a) \cdot \rho_{j,t}(b), t)] + [(\rho_{i,t}(a) \cdot \rho_{k,t}(c), t)] \\ &= [(a,i)] \cdot [(b,j)] + [(a,i)] \cdot [(c,k)]\end{aligned}$$

Hence $(\cdot)$ distributes over $(+)$.

Thus $\left(\varinjlim A_i, +, \cdot\right)$ is a ring.

Comment

Step 12 of 16

However, we have the following.

(1) Consider the A_i are all commutative. Suppose $[(a,i)], [(b,j)] \in \varinjlim A_i$, and let $k \ge i, j$. Then we have the following:

$$\begin{aligned}[(a,i)] \cdot [(b,j)] &= [(\rho_{i,k}(a) \cdot \rho_{j,k}(b), k)] \\ &= [(\rho_{j,k}(b) \cdot \rho_{i,k}(a), k)] \\ &= [(b,j)] \cdot [(a,i)]\end{aligned}$$

Hence $\varinjlim A_i$ is a commutative ring. If all the A_i are commutative, then $\varinjlim A_i$ is commutative.

(2) Note that because the $\rho_{i,j}$ are unital ring homomorphism

$$\begin{aligned}\rho_{i,k}(1) &= 1 \\ &= \rho_{j,k}(1)\end{aligned}$$

Whenever $k \ge i, j$. Hence $[(1,i)] = [(1,j)]$ for all i, j. Define $1 = [(1,i)]$.

Consider $[(a,i)] \in \varinjlim A_i$. Then ,

$$\begin{aligned}1 \cdot [(a,i)] &= [(1,i)] \cdot [(a,i)] \\ &= [\rho_{i,i}(1) \cdot \rho_{i,i}(a), i] \\ &= [(1 \cdot a, i)] \\ &= [(a,i)]\end{aligned}$$

Thus, $[(a,i)] \cdot 1 = [(a,i)]$. Hence 1 is a multiplicative identity in $\varinjlim A_i$. If all the A_i have $1 \ne 0$ and the $\rho_{i,j}$ are unital, then $\varinjlim A_i$ has a multiplicative identity.

3. Suppose $[(0,i)] = [(1,i)]$. Then there exists $j \ge i$ such that $\rho_{i,j}(0) = \rho_{i,j}(1)$, so that $0 = 1$ in A_j a contradiction. Hence $1 \ne 0$ in $\varinjlim A_i$.

Thus if the A_i are commutative rings with $1 \ne 0$, then $\varinjlim A_i$ is a commutative ring with $1 \ne 0$. Thus, if $1 \ne 0$ for all A_i, then $1 \ne 0$ in $\varinjlim A_i$.

Comment

Step 13 of 16

Therefore, that if all the A_i are commutative rings with $1 \ne 0$ and all the $\rho_{i,j}$ are unital ring homomorphism, then $\varinjlim A_i$ may likewise be given the structure of a commutative ring with $1 \ne 0$ such that the ρ_i are all ring homomorphism.

Comment

Step 14 of 16

(e) Under the hypotheses of part (c), prove that $\varinjlim A_i$ has the following universal property: if *C* is any abelian group such that for each $i \in I$ there is a homomorphism $\varphi_i : A_i \longrightarrow C$ with $\varphi_i = \varphi_j \circ \rho_{i,j}$ whenever $i \le j$, then there is a unique homomorphism $\varphi : A \longrightarrow C$ such that $\varphi \circ \rho_i = \varphi_i$ for all *i*.

Consider, *C* is an abelian group and that we have an indexed family of group homomorphism $\varphi_i : A_i \longrightarrow C$ such that $\varphi_i = \varphi_j \circ \rho_{i,j}$ for all $i, j \in I$.

Define $\varphi : \varinjlim A_i \longrightarrow C$ by $\varphi([(a,i)]) = \varphi_i(a)$. We need to show that φ is a well defined group homomorphism.

Suppose $[(a,i)] = [(b,j)]$.

Then there exists $k \ge i, j$ such that $\rho_{i,k}(a) = \rho_{j,k}(b)$. Since $\rho_{i,j}(s)$ is well defined, $\varphi_k(\rho_{i,k}(a)) = \varphi_k(\rho_{j,k}(b))$. Then, $(\varphi_k \circ \rho_{i,k})(a) = (\varphi_k \circ \rho_{j,k})(b)$, and we have $\varphi_i(a) = \varphi_j(b)$. Hence, $\varphi([(a,i)]) = \varphi([b,j])$, and φ is well defined.

Comment

Step 15 of 16

Consider $[(a,i)], [(b,j)] \in \varinjlim A_i$, and let $k \ge i, j$. Then we have the following:

$$\begin{aligned}\varphi([(a,i)] + [(b,i)]) &= \varphi([(\rho_{i,k}(a) + \rho_{j,k}(b), k)]) \\ &= \varphi_k(\rho_{i,k}(a) + \rho_{j,k}(b)) \\ &= \varphi_k(\rho_{i,k}(a)) + \varphi_k(\rho_{j,k}(b)) \\ &= (\varphi_k \circ \rho_{i,k})(a) + (\varphi_k \circ \rho_{j,k})(b)\end{aligned}$$

$$\begin{aligned}\varphi([(a,i)] + [(b,i)]) &= \varphi_i(a) + \varphi_j(b) \\ &= \varphi([(a,i)]) + \varphi([(b,j)])\end{aligned}$$

So φ is a group homomorphism. Finally, for all *i* and all $a \in A_i$,

$$\begin{aligned}(\varphi \circ \rho_i)(a) &= \varphi(\rho_i(a)) \\ &= \varphi([(a,i)]) \\ &= \varphi_i(a)\end{aligned}$$

Hence, $\varphi \circ \rho_i = \varphi_i$ for all *i*.

Comment

Step 16 of 16

Suppose that we have a group homomorphism $\psi : \varinjlim A_i \longrightarrow C$ which also satisfies $\psi \circ \rho_i = \varphi_i$ for all $i \in I$. Then for all *i* and all $a \in A_i$,

$$\begin{aligned}\psi([(a,i)]) &= \psi(\rho_i(a)) \\ &= (\psi \circ \rho_i)(a) \\ &= \varphi_i(a) \\ &= (\varphi \circ \rho_i)(a)\end{aligned}$$

$$\begin{aligned}\psi([(a,i)]) &= \varphi(\rho_i(a)) \\ &= \varphi([(a,i)])\end{aligned}$$

Hence $\psi = \varphi$.

Therefore, $\varinjlim A_i$ has the following universal property: if *C* is any abelian group such that for each $i \in I$ there is a homomorphism $\varphi_i : A_i \longrightarrow C$ with $\varphi_i = \varphi_j \circ \rho_{i,j}$ whenever $i \le j$, then there is a unique homomorphism $\varphi : A \longrightarrow C$ such that $\varphi \circ \rho_i = \varphi_i$ for all *i*.

Comment