

# Abstract Algebra (3rd Edition)

Chapter 1.4, Problem 10E (2 Bookmarks) Show all steps:  on

## Step-by-step solution

### Step 1 of 12

(a) Now first we compute matrix product  $XY$ . Here,  $X, Y \in H(F)$ .  
Therefore,

$$XY = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+d & b+af+e \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{pmatrix}$$

So that  $XY \in H(F)$ .  
Therefore,  $H(F)$  is closed under matrix multiplication.

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### Step 2 of 12

Suppose  $F$  is a field and  $0, 1 \in F$ . Let us consider:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in H(F)$$

Now prove that  $AB \neq BA$ .

On computing  $AB$ , we have:

$$AB = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

On computing  $BA$ , we have:

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,  $AB \neq BA$ , so that  $H(F)$  is non-abelian for all  $F$ .

(b) To find an explicit for the matrix inverse  $X^{-1}$  and then show that  $H(F)$  is closed under inverses. Let us consider  $X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ .

Find  $X^{-1}$  such that  $XX^{-1} = I$ . Therefore,

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,  $X^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \in H(F)$

Hence  $H(F)$  is closed under matrix inverses.

(c) To prove the associative law for  $H(F)$  then show that  $H(F)$  is group of order  $|F|^3$ .

Let us consider  $X, Y, Z \in H(F)$ , such that:

$$X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}$$

Now show that  $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$

Compute Right Hand Side of the above equation.

$$\begin{aligned} X \cdot (Y \cdot Z) &= \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d+g & e+hf+e \\ 0 & 1 & f+i \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a+d+g & e+hf+e+ai+b \\ 0 & 1 & c+f+i \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a+d & e+af+b \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix} \\ &= (X \cdot Y) \cdot Z \end{aligned}$$

Therefore, matrix multiplication is associative on  $H(F)$ .

Also  $H(F)$  is group under matrix multiplication.

Therefore, it is proved that  $|H(F)| = |F|^3$ .

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### Step 3 of 12

(d) To find the order of each element of the finite group  $H(\mathbb{Z}/2\mathbb{Z})$ .

First list all the possible elements of the finite group  $H(\mathbb{Z}/2\mathbb{Z})$  as follows:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_7 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_8 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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### Step 4 of 12

Now find the order of each element of the finite group  $H(\mathbb{Z}/2\mathbb{Z})$ .

Compute:

$$A_1^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_1$  is 2.

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### Step 5 of 12

Compute:

$$A_2^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_2$  is 2.

Compute:

$$A_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_3$  is 2.

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### Step 6 of 12

Compute:

$$A_4^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_4$  is 2.

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### Step 7 of 12

Compute:

$$A_5^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_5$  is 2.

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### Step 8 of 12

Compute:

$$A_6^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_7^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_6$  is 3.

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### Step 9 of 12

Compute:

$$A_8^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, the order of  $A_8$  is 4.

(e) To prove that the every non-identity element of the group  $H(\mathbb{Z})$  has infinite order.

Let us consider that all  $a, b, c \in \mathbb{Z}$  and also  $n \geq 1$

Show that:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na & nb+ta,ac \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } t_k = \sum_{i=0}^{k-1} i$$

We prove the above by using induction on  $n$ .

First we prove for  $n=1$ .

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot a & 1b+t_0,ac \\ 0 & 1 & 1 \cdot c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot a & 1b+0 \\ 0 & 1 & 1 \cdot c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

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If this is true for  $n=1$ , then we assume that it is true for  $n+1$ .

So,

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & na & nb+t_n,ac \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (n+1)a & (n+1)b+t_n,ac \\ 0 & 1 & (n+1)c \\ 0 & 0 & 1 \end{pmatrix}$$

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Now we consider any arbitrary element  $X \in H(\mathbb{Z})$ .

$$\text{Where } X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

If either  $a$  or  $c$  is non-zero, then  $(1, 2)$  or  $(2, 3)$  are the non-zero element of  $X^n$  for all positive  $n$ .

If  $a=c=0$  then  $b \neq 0$ .

Therefore,  $(1, 3)$  is the non-zero element of  $X^n$  for all positive  $n$ .

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### Step 12 of 12

Therefore, every non-identity element of the group  $H(\mathbb{Z})$  has infinite order.

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