Abstract Algebra (3rd Edition)

Chapter 1.4, Problem 10E

Step-by-step solution

Step 1 of 12

(a) Now first we compute matrix product XY. Here, $X, Y \in H(F)$.

Therefore,

$$XY = \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+d & b+af+e\\0 & 1 & c+f\\0 & 0 & 1\end{pmatrix}$$

So that $XY \in H(F)$.

Therefore, $H(F)$ is closed under matrix multiplication.

Comment

Step 2 of 12

Suppose F is a field and $0, 1 \in F$. Let us consider:

$$A = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \text{ and } B = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

Now prove that $AB \neq BA$.

On computing AB, we have:

$$AB = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

On computing BA, we have:

$$BA = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

Therefore, $AB \neq BA$, so that $H(F)$ is non-abelian for all F.

(b) To find an explicit for the matrix inverse X^{-1} and then show that $H(F)$ is closed under inverses. Let us consider $X = \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}$

First, X^{-1} such that $XX^{-1} = I$. Therefore,

$$\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, $X^{-1} = \begin{pmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{pmatrix} \in H(F)$

Hence $H(F)$ is closed under matrix inverses.

(c) To prove the associative low for $H(F)$ then show that $H(F)$ is group of order $|F|^3$.

Let us consider $X, Y, Z \in H(F)$, such that:

$$X = \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}$$

$$Y = \begin{pmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{pmatrix}$$

$$Z = \begin{pmatrix}1 & g & h\\0 & 1 & i\\0 & 0 & 1\end{pmatrix}$$

Now show that $X\cdot(Y\cdot Z) = (X\cdot Y)\cdot Z$

Compute Right Hand Side of the above equation.

$$\begin{aligned}X\cdot(Y\cdot Z) &= \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & g & h\\0 & 1 & i\\0 & 0 & 1\end{pmatrix}\right)\\ &= \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & d+g & h+di+e\\0 & 1 & f+i\\0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & a+d+g & h+di+e+af+ai+b\\0 & 1 & c+f+i\\0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & a+d & e+af+b\\0 & 1 & c+f\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & g & h\\0 & 1 & i\\0 & 0 & 1\end{pmatrix}\end{aligned}$$

$$= \left(\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{pmatrix}\right)\cdot\begin{pmatrix}1 & g & h\\0 & 1 & i\\0 & 0 & 1\end{pmatrix}$$

$$= (X\cdot Y)\cdot Z$$

Therefore, matrix multiplication is associative on $H(F)$.

Also $H(F)$ is group under matrix multiplication.

Therefore, it is proved that $|H(F)| = |F|^3$.

Comments (2)

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(d) To find the order of each element of the finite group $H(\mathbb{Z}/2\mathbb{Z})$.

First list all the possible elements of the finite group $H(\mathbb{Z}/2\mathbb{Z})$ as follows:

$$A_1 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, A_2 = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, A_3 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, A_4 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

$$A_5 = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, A_6 = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}, A_7 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}, A_8 = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

Comment

Step 4 of 12

Now find the order of each element of the finite group $H(\mathbb{Z}/2\mathbb{Z})$.

Compute:

$$A_2^2 = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_2 is 2.

Comment

Step 5 of 12

Compute:

$$A_3^2 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_3 is 2.

Compute:

$$A_4^2 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_4 is 2.

Comment

Step 6 of 12

Compute:

$$A_5^2 = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_5 is 2.

Comment

Step 7 of 12

Compute:

$$A_7^2 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_7 is 2.

Comment

Step 8 of 12

Compute:

$$A_6^2 = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

$$A_6^3 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_6 is 3.

Comments (2)

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Compute:

$$A_8^2 = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

$$A_8^3 = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

$$A_8^4 = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Therefore, the order of A_8 is 4.

(e) To prove that the every non-identity element of the group $H(\mathbb{Z})$ has infinite order.

Let us consider that all $a, b, c \in \mathbb{Z}$ and also $n \geq 1$

Show that:

$$\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & na & nb+t_{n-1}ac\\0 & 1 & nc\\0 & 0 & 1\end{pmatrix}, \text{ where } t_k = \sum_{i=0}^{k} i$$

We prove the above by using induction on *n*.

First we prove for $n = 1$.

$$\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}^1 = \begin{pmatrix}1 & 1\cdot a & 1.b+t_{1-1}ac\\0 & 1 & 1\cdot c\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\cdot a & 1.b+0\\0 & 1 & 1\cdot c\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}$$

Comment

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If this is true for $n = 1$, then we assume that it is true for $n+1$.

So,

$$\begin{aligned}\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}^{n+1} &= \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}^n\\ &= \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & na & nb+t_{1-1}ac\\0 & 1 & nc\\0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & (n+1)a & (n+1)b+t_nac\\0 & 1 & (n+1)c\\0 & 0 & 1\end{pmatrix}\end{aligned}$$

Comment

Step 11 of 12

Now we consider any arbitrary element $X \in H(\mathbb{Z})$.

Where $X = \begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}$

If either *a* or *c* is non-zero, then $(1,2)$ or $(2,3)$ are the non-zero element of X^n for all positive *n*.

If $a = c = 0$ then $b \neq 0$.

Therefore, $(1,3)$ is the non-zero element of X^n for all positive *n*.

Comment

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Therefore, every non-identity element of the group $H(\mathbb{Z})$ has infinite order.

Comment