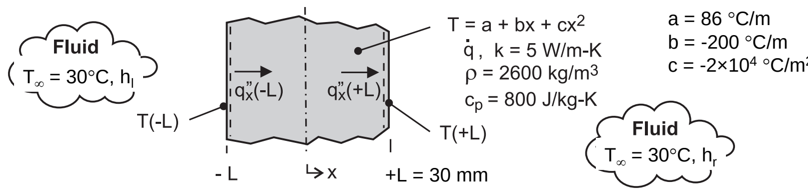
PROBLEM 2.32

KNOWN: Analytical expression for the steady-state temperature distribution of a plane wall experiencing uniform volumetric heat generation $\dot{q}$ while convection occurs at both of its surfaces.

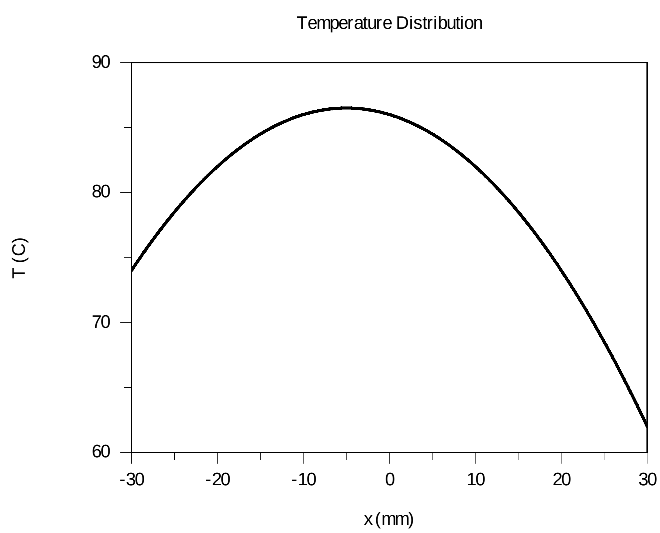
FIND: (a) Sketch the temperature distribution, T(x), and identify significant physical features, (b) Determine $\dot{q}$, (c) Determine the surface heat fluxes, $q''_x(-L)$ and $q''_x(+L)$; how are these fluxes related to the generation rate; (d) Calculate the convection coefficients at the surfaces x = L and x = +L, (e) Obtain an expression for the heat flux distribution, $q''_x(x)$; explain significant features of the distribution; (f) If the source of heat generation is suddenly deactivated ($\dot{q}$ = 0), what is the rate of change of energy stored at this instant; (g) Determine the temperature that the wall will reach eventually with $\dot{q} = 0$; determine the energy that must be removed by the fluid per unit area of the wall to reach this state.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Uniform volumetric heat generation, (3) Constant properties.

ANALYSIS: (a) Using the analytical expression in the Workspace of IHT, the temperature distribution appears as shown below. The significant features include (1) parabolic shape, (2) maximum does not occur at the mid-plane, T(-5 mm) = 86.5°C, (3) the gradient at the x = +L surface is greater than at x = -L. Find also that T(-L) = 74°C and T(+L) = 62°C for use in part (d).



(b) Substituting the temperature distribution expression into the appropriate form of the heat diffusion equation, Eq. 2.21, the rate of volumetric heat generation can be determined.

$$\frac{d}{dx}\left(\frac{dT}{dx}\right)+\frac{\dot{q}}{k}=0 \qquad \text{where} \qquad T(x)=a+bx+cx^2$$

$$\frac{d}{dx}(0+b+2cx)+\frac{\dot{q}}{k}=(0+2c)+\frac{\dot{q}}{k}=0$$

Continued ...

PROBLEM 2.32 (Cont.)

$$\dot{q} = -2ck = -2\left(-2\times10^{4}\,°\mathrm{C/m^2}\right)5\,\mathrm{W/m\cdot K} = 2\times10^{5}\,\mathrm{W/m^3}$$ <

(c) The heat fluxes at the two boundaries can be determined using Fourier's law and the temperature distribution expression.

$$q''_x(x) = -k\frac{dT}{dx} \qquad \text{where} \qquad T(x) = a + bx + cx^2$$

$$q''_x(-L) = -k\left[0 + b + 2cx\right]_{x=-L} = -\left[b - 2cL\right]k$$

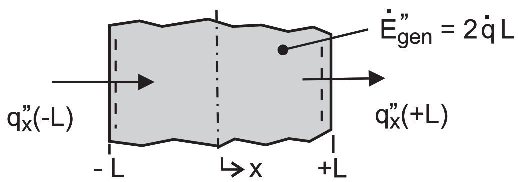
$$q''_x(-L) = -\left[-200°\mathrm{C/m} - 2\left(-2\times10^{4}\,°\mathrm{C/m^2}\right)0.030\mathrm{m}\right]\times 5\,\mathrm{W/m\cdot K} = -5000\,\mathrm{W/m^2}$$ <

$$q''_x(+L) = -\left(b + 2cL\right)k = +7000\,\mathrm{W/m^2}$$ <

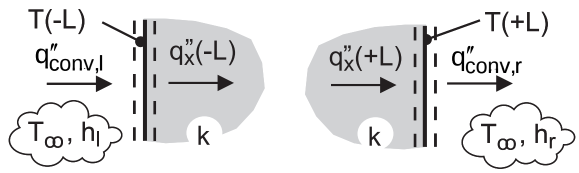
From an overall energy balance on the wall as shown in the sketch below, $\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = 0$,

$$+q''_x(-L) - q''_x(+L) + 2\dot{q}L \stackrel{?}{=} 0 \quad \text{or} \quad -5000\,\mathrm{W/m^2} - 7000\,\mathrm{W/m^2} + 12{,}000\,\mathrm{W/m^2} = 0$$

where $2\dot{q}L = 2\times2\times10^{5}\,\mathrm{W/m^3}\times0.030\ \mathrm{m} = 12{,}000\,\mathrm{W/m^2}$, so the equality is satisfied



Part (c) Overall energy balance



Part (d) Surface energy balances

(d) The convection coefficients, h_l and h_r, for the left- and right-hand boundaries (x = -L and x= +L, respectively), can be determined from the convection heat fluxes that are equal to the conduction fluxes at the boundaries. See the surface energy balances in the sketch above. See also part (a) result for T(-L) and T(+L).

$$q''_{conv,\ell} = q''_x(-L)$$

$$h_l\left[T_\infty - T(-L)\right] = h_l\left[30 - 74\right]\mathrm{K} = -5000\,\mathrm{W/m^2} \qquad h_l = 114\,\mathrm{W/m^2\cdot K}$$ <

$$q''_{conv,r} = q''_x(+L)$$

$$h_r\left[T(+L) - T_\infty\right] = h_r\left[62 - 30\right]\mathrm{K} = +7000\,\mathrm{W/m^2} \qquad h_r = 219\,\mathrm{W/m^2\cdot K}$$ <

(e) The expression for the heat flux distribution can be obtained from Fourier's law with the temperature distribution

$$q''_x(x) = -k\frac{dT}{dx} = -k\left[0 + b + 2cx\right]$$

$$q''_x(x) = -5\,\mathrm{W/m\cdot K}\left[-200°\mathrm{C/m} + 2\left(-2\times10^{4}\,°\mathrm{C/m^2}\right)\right]x = 1000 + 2\times10^{5}x$$ <

Continued …

The distribution is linear with the x-coordinate. The maximum temperature will occur at the location where $q''_x(x_{max}) = 0$,

$$x_{max} = -\frac{1000\,W/m^2}{2\times10^5\ W/m^3} = -5.00\times10^{-3}m = -5\,mm \quad <$$

(f) If the source of the heat generation is suddenly deactivated so that $\dot{q}$ = 0, the appropriate form of the heat diffusion equation for the ensuing transient conduction is

$$k\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \rho c_p \frac{\partial T}{\partial t}$$

At the instant this occurs, the temperature distribution is still $T(x) = a + bx + cx^2$. The right-hand term represents the rate of energy storage per unit volume,

$$\dot{E}''_{st} = k\frac{\partial}{\partial x}[0+b+2cx] = k[0+2c] = 5\,W/m\cdot K\times2\left(-2\times10^{4\circ}C/m^2\right) = -2\times10^5\ W/m^3 \quad <$$

(g) With no heat generation, the wall will eventually ($t \to \infty$) come to equilibrium with the fluid, $T(x,\infty) = T_\infty = 30°C$. To determine the energy that must be removed from the wall to reach this state, apply the conservation of energy requirement over an interval basis, Eq. 1.12b. The "initial" state is that corresponding to the steady-state temperature distribution, T_i, and the "final" state has $T_f = 30°C$. We've used T_∞ as the reference condition for the energy terms.

$$E''_{in} - E''_{out} = \Delta E''_{st} = E''_f - E''_i \qquad \text{with} \quad E''_{in} = 0.$$

$$E''_{out} = \rho c_p \int_{-L}^{+L} \left(T_i - T_\infty\right) dx$$

$$E''_{out} = \rho c_p \int_{-L}^{+L} \left[a + bx + cx^2 - T_\infty\right] dx = \rho c_p \left[ax + bx^2/2 + cx^3/3 - T_\infty x\right]_{-L}^{+L}$$

$$E''_{out} = \rho c_p \left[2aL + 0 + 2cL^3/3 - 2T_\infty L\right]$$

$$E''_{out} = 2600\,kg/m^3 \times 800\,J/kg\cdot K\left[2\times86°C\times0.030m + 2\left(-2\times10^{4\circ}C/m^2\right)\right.$$

$$\left.(0.030m)^3/3 - 2(30°C)0.030m\right]$$

$$E''_{out} = 6.24\times10^6\,J/m^2 \quad <$$

COMMENTS: (1) In part (a), note that the temperature gradient is larger at x = + L than at x = - L. This is consistent with the results of part (c) in which the conduction heat fluxes are evaluated.

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PROBLEM 2.32 (Cont.)

(2) In evaluating the conduction heat fluxes, $q''_x(x)$, it is important to recognize that this flux is in the positive x-direction. See how this convention is used in formulating the energy balance in part (c).

(3) It is good practice to represent energy balances with a schematic, clearly defining the system or surface, showing the CV or CS with dashed lines, and labeling the processes. Review again the features in the schematics for the energy balances of parts (c & d).

(4) Re-writing the heat diffusion equation introduced in part (b) as

$$-\frac{d}{dx}\left(-k\frac{dT}{dx}\right)+\dot{q}=0$$

recognize that the term in parenthesis is the heat flux. From the differential equation, note that if the differential of this term is a constant $(\dot{q}/k)$, then the term must be a linear function of the x-coordinate. This agrees with the analysis of part (e).

(5) In part (f), we evaluated $\dot{E}_{st}$, the rate of energy change stored in the wall at the instant the volumetric heat generation was deactivated. Did you notice that $\dot{E}_{st} = -2\times10^5\ W/m^3$ is the same value of the deactivated $\dot{q}$? How do you explain this?