

Chapter 1

General background

1.1 Optical fields and Maxwell's equations

1.1.1 Maxwell's equations and the continuity equation are:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t}.\end{aligned}$$

From Section 1.1, we know that \mathbf{E} , \mathbf{D} , and ∇ all change sign under spatial inversion but not under time reversal. We also know that \mathbf{H} , \mathbf{B} , and $\partial/\partial t$ all change sign under time reversal but not under spatial inversion. \mathbf{J} changes sign under either spatial inversion or time reversal, but ρ never changes sign under spatial inversion or time reversal.

(a) Taking spatial inversion, the equations become:

$$\begin{aligned}-\nabla \times (-\mathbf{E}) &= -\frac{\partial \mathbf{B}}{\partial t}, \\ -\nabla \times \mathbf{H} &= -\mathbf{J} + \frac{\partial(-\mathbf{D})}{\partial t}, \\ -\nabla \cdot \mathbf{B} &= 0, \\ -\nabla \cdot (-\mathbf{D}) &= \rho, \\ -\nabla \cdot (-\mathbf{J}) &= -\frac{\partial \rho}{\partial t}.\end{aligned}$$

Each equation returns to its original form after the signs are cleared up. Hence, Maxwell's equations and the continuity equation are invariant under spatial inversion.

(b) Taking time reversal, the equations become:

$$\nabla \times \mathbf{E} = -\frac{\partial(-\mathbf{B})}{\partial(-t)},$$

$$\begin{aligned}\nabla \times (-\mathbf{H}) &= -\mathbf{J} + \frac{\partial \mathbf{D}}{\partial(-t)}, \\ \nabla \cdot (-\mathbf{B}) &= 0, \\ \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot (-\mathbf{J}) &= -\frac{\partial \rho}{\partial(-t)}.\end{aligned}$$

Each equation returns to its original form after the signs are cleared up. Hence, Maxwell's equations and the continuity equation are invariant under time reversal.

(c) Taking both spatial inversion and time reversal, the equations become:

$$\begin{aligned}-\nabla \times (-\mathbf{E}) &= -\frac{\partial(-\mathbf{B})}{\partial(-t)}, \\ -\nabla \times (-\mathbf{H}) &= \mathbf{J} + \frac{\partial(-\mathbf{D})}{\partial(-t)}, \\ -\nabla \cdot (-\mathbf{B}) &= 0, \\ -\nabla \cdot (-\mathbf{D}) &= \rho, \\ -\nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial(-t)}.\end{aligned}$$

Note the sign of \mathbf{J} is not changed here because it changes twice in this situation. Each equation returns to its original form after the signs are cleared up. Hence, Maxwell's equations and the continuity equation are invariant under simultaneous spatial inversion and time reversal.

1.2 Harmonic fields

1.3 Linear optical susceptibility

1.3.1 By definition,

$$\chi(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} d\mathbf{r} \int_{-\infty}^{\infty} dt \chi(\mathbf{r}, t) \exp(-i\mathbf{k} \cdot \mathbf{r} + i\omega t).$$

Because $\chi(\mathbf{r}, t)$ is always real, $\chi(\mathbf{r}, t) = \chi(\mathbf{r}, t)^*$. Then,

$$\begin{aligned}\chi^*(\mathbf{k}, \omega) &= \int_{-\infty}^{\infty} d\mathbf{r} \int_{-\infty}^{\infty} dt \chi^*(\mathbf{r}, t) \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) \\ &= \int_{-\infty}^{\infty} d\mathbf{r} \int_{-\infty}^{\infty} dt \chi(\mathbf{r}, t) \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) \\ &= \chi(-\mathbf{k}, -\omega).\end{aligned}$$

Therefore, $\chi_{ij}^*(\mathbf{k}, \omega) = \chi_{ij}(-\mathbf{k}, -\omega)$. Similarly, $\epsilon_{ij}^*(\mathbf{k}, \omega) = \epsilon_{ij}(-\mathbf{k}, -\omega)$.

1.4 Polarization of light

1.4.1 Let the optical beam propagate along the z direction. Then its electric field lies in the xy plane and can be generally expressed as

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}(\mathbf{r}, t) \exp(ikz - i\omega t) \quad \text{with} \quad \mathcal{E} = \hat{x}\mathcal{E}_x + \hat{y}\mathcal{E}_y.$$

(a) The transmission axes of the cross polarizers are orthogonal to each other. Without loss of generality, the axis of the first polarizer at the input is taken to be in the x direction and that of the second polarizer at the output is in the y direction. The electric field after the first polarizer is

$$\mathcal{E}_1 = \hat{x}(\hat{x} \cdot \mathcal{E}) = \hat{x}\mathcal{E}_x.$$

Because the second polarizer only allows the y component of the electric field to pass, the electric field after the second polarizer is

$$\mathcal{E}_2 = \hat{y}(\hat{y} \cdot \mathcal{E}_1) = \hat{y}(\hat{y} \cdot \hat{x}\mathcal{E}_x) = 0.$$

Thus no light is transmitted through the cross polarizers.

(b) The transmission axis of the inserted polarizer is generally placed in a direction defined by the unit vector $\hat{x}' = \hat{x} \cos \theta + \hat{y} \sin \theta$. This polarizer is now marked as the second polarizer, while the output cross polarizer that has an axis in the y direction and is marked as the second polarizer in (a) is now marked as the third polarizer. The electric field after the first polarizer at the input is

$$\mathcal{E}_1 = \hat{x}(\hat{x} \cdot \mathcal{E}) = \hat{x}\mathcal{E}_x,$$

as in (a). The electric field after the inserted second polarizer here is

$$\mathcal{E}_2 = \hat{x}'(\hat{x}' \cdot \mathcal{E}_1) = \hat{x}\mathcal{E}_x \cos^2 \theta + \hat{y}\mathcal{E}_x \cos \theta \sin \theta.$$

The electric field after the third polarizer at the output here is now

$$\mathcal{E}_3 = \hat{y}(\hat{y} \cdot \mathcal{E}_2) = \hat{y}\mathcal{E}_x \cos \theta \sin \theta.$$

Thus the transmittance through all three polarizers from the input to the output is

$$T = \frac{|\mathcal{E}_3|^2}{|\mathcal{E}|^2} = \frac{|\mathcal{E}_x|^2}{4(|\mathcal{E}_x|^2 + |\mathcal{E}_y|^2)} \sin^2 2\theta.$$

(c) A polarizer of the transmission type preferentially transmits only the field component along its transmission axis. The orthogonal field component is block by absorption or reflection. In the case of (a), the field incident on the output polarizer contains only a component that is orthogonal to the transmission axis of the polarizer and is thus blocked. In the case of (b), the same field that is polarized orthogonally to the transmission axis of the output polarizer is first incident on the inserted polarizer. What comes out of this polarizer is a field that is no longer polarized orthogonally to the transmission axis of the output polarizer. Thus there is a finite transmittance that depends on the angle of the axis of the inserted polarizer with respect to those of the cross polarizers. The key to this entire process is the fact that the inserted polarizer transmits a field component that is not parallel to the polarization of the incident field if the polarizer axis is not parallel or orthogonal to the incident field polarization. This is a manifestation of the vector nature of an optical field.

1.5 Propagation in an isotropic medium

1.5.1 By definition, $k = \omega^2 \mu_0 \epsilon$ and $\epsilon = \epsilon_0(1 + \chi)$. In the presence of an optical loss or gain, $\epsilon = \epsilon' + i\epsilon''$ and $\chi = \chi' + i\chi''$. In the case when $\chi'' \ll \chi'$, the refractive index is

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 + \chi' + i\chi''} \approx \sqrt{1 + \chi'}.$$

Then,

$$\begin{aligned} k &= \omega \sqrt{\mu_0 \epsilon} \\ &= \omega \sqrt{\mu_0 \epsilon_0} \sqrt{1 + \chi' + i\chi''} \\ &= \omega \sqrt{\mu_0 \epsilon_0} \sqrt{n^2 + i\chi''}, \\ &\approx \omega \sqrt{\mu_0 \epsilon_0} \left(n + i \frac{\chi''}{2n} \right). \end{aligned}$$

Because

$$k = \beta + i \frac{\alpha}{2}$$

from (1.100), we have

$$\beta \approx n\omega \sqrt{\mu_0 \epsilon_0}, \quad \alpha \approx \frac{\chi''}{n} \omega \sqrt{\mu_0 \epsilon_0}.$$

Thus,

$$\alpha \approx \beta \frac{\chi''}{n^2}.$$

1.5.2 With the given parameters, $\chi' = 15.48$ and $\chi'' = 0.284$. Thus, $\chi'' \ll \chi'$. We find the following characteristics for silicon at 532 nm wavelength:

$$\begin{aligned} n &= \sqrt{1 + \chi'} = \sqrt{1 + 15.48} = 4.06, \\ \beta &= \frac{2n\pi}{\lambda} = \frac{2 \times 4.06 \times \pi}{532 \times 10^{-9}} \text{ m}^{-1} = 4.8 \times 10^7 \text{ m}^{-1}, \\ \alpha &= \beta \frac{\chi''}{n^2} = 4.8 \times 10^7 \times \frac{0.284}{4.06^2} \text{ m}^{-1} = 8.26 \times 10^5 \text{ m}^{-1}. \end{aligned}$$

(a) Because $n_1 = 1$ for the air and $n_2 = 4.06$ for silicon, the surface reflectivity is

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2 = \left(\frac{4.06 - 1}{4.06 + 1} \right)^2 = 36.6\%.$$

Thus, for an incident power of 1 W, the reflected light has $P_r = 366 \text{ mW}$, and the light entering the silicon wafer has $P_{\text{in}} = 634 \text{ mW}$.

(b) For $l = 100 \text{ } \mu\text{m}$, $\alpha l = 82.6$. Hence the transmitted power is

$$P_t = P_{\text{in}} e^{-\alpha l} = 634 \text{ mW} \times e^{-82.6} = 8.5 \times 10^{-37} \text{ W} \approx 0.$$

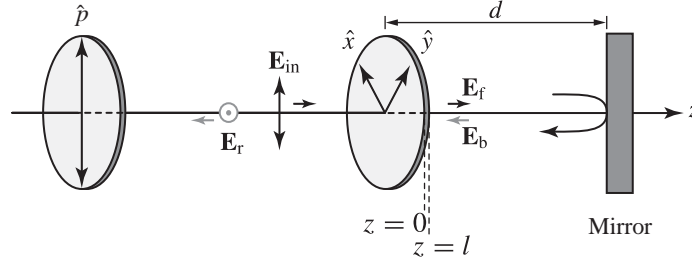
Practically no light is transmitted.

(c) For $P_t = 1 \text{ mW}$, the thickness l is found as

$$1 = 634 \times \exp(-8.26 \times 10^5 \times l) \quad \Rightarrow \quad l = \frac{\ln 634}{8.26 \times 10^5} \text{ m} = 7.8 \text{ } \mu\text{m}.$$

1.6 Propagation in an anisotropic medium

1.6.1



Take the optical wave to propagate in the z direction with $z = 0$ at the front surface of the quarter-wave plate, the principal axes of the quarter-wave plate to be \hat{x} and \hat{y} , the thickness of the quarter-wave plate to be $l = l_{\lambda/4}$, and the distance between the quarter-wave plate and the mirror to be d , as shown. The axis of the polarizer is in the direction $\hat{p} = (\hat{x} + \hat{y})/\sqrt{2}$. Assume that $k^y > k^x$ so that $(k^y - k^x)l = (k^y - k^x)l_{\lambda/4} = \pi/2$. Then, the polarization states of the optical wave at different positions are

linearly polarized along \hat{p} before entering quarter-wave plate:

$$\mathbf{E}_{\text{in}}(z = 0, t) = \hat{p}\mathcal{E}e^{-i\omega t} = \frac{\hat{x} + \hat{y}}{\sqrt{2}}\mathcal{E}e^{-i\omega t},$$

circularly polarized after first pass through quarter-wave plate:

$$\begin{aligned}\mathbf{E}_{\text{f}}(z = l, t) &= \left(\hat{x}e^{ik^x l} + \hat{y}e^{ik^y l} \right) \frac{\mathcal{E}}{\sqrt{2}}e^{-i\omega t} \\ &= (\hat{x} + i\hat{y}) \frac{\mathcal{E}}{\sqrt{2}}e^{i(k^x l - \omega t)},\end{aligned}$$

circularly polarized before and after mirror back to quarter-wave plate:

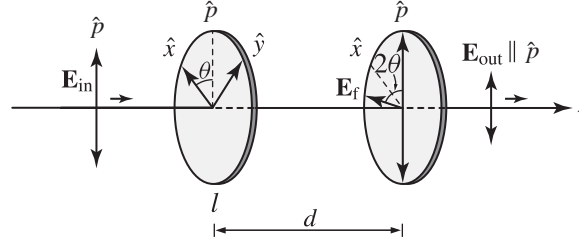
$$\mathbf{E}_{\text{b}}(z = l, t) = \mathbf{E}_{\text{f}}(z = l, t)e^{ik(2d) + i\varphi} = (\hat{x} + i\hat{y}) \frac{\mathcal{E}}{\sqrt{2}}e^{i(k^x l + 2kd - \omega t + \varphi)},$$

linearly polarized after second pass through quarter-wave plate:

$$\begin{aligned}\mathbf{E}_{\text{r}}(z = 0, t) &= \left(\hat{x}e^{ik^x l} + i\hat{y}e^{ik^y l} \right) \frac{\mathcal{E}}{\sqrt{2}}e^{i(k^x l + 2kd - \omega t + \varphi)} \\ &= (\hat{x} - \hat{y}) \frac{\mathcal{E}}{\sqrt{2}}e^{i(2k^x l + 2kd - \omega t + \varphi)},\end{aligned}$$

where φ is the phase on reflection from the mirror. Because $(\hat{x} - \hat{y}) \cdot \hat{p} = (\hat{x} - \hat{y}) \cdot (\hat{x} + \hat{y})/\sqrt{2} = 0$, the return field is linearly polarized in a direction that is perpendicular to the polarizer axis and thus is blocked by the polarizer.

1.6.2



Take the optical wave to propagate in the z direction with $z = 0$ at the front surface of the half-wave plate, the principal axes of the half-wave plate to be \hat{x} and \hat{y} , the thickness of the half-wave plate to be $l = l_{\lambda/2}$, and the distance between the half-wave plate and the polarizer at the output to be d , as shown. The axis of the polarizer is in the direction $\hat{p} = \hat{x} \cos \theta + \hat{y} \sin \theta$, which makes an angle θ with respect to the \hat{x} axis of the half-wave plate. Assume that $k^y > k^x$ so that $(k^y - k^x)l = (k^y - k^x)l_{\lambda/2} = \pi$. The input optical field is linearly polarized in a direction parallel to \hat{p} . Then, the polarization states of the optical wave at different positions are

linearly polarized along \hat{p} before entering half-wave plate:

$$\mathbf{E}_{in}(z = 0, t) = \hat{p} \mathcal{E} e^{-i\omega t} = (\hat{x} \cos \theta + \hat{y} \sin \theta) \mathcal{E} e^{-i\omega t},$$

linearly polarized at another direction after passing through half-wave plate:

$$\begin{aligned} \mathbf{E}_f(z = l, t) &= (\hat{x} \cos \theta e^{ik^x l} + \hat{y} \sin \theta e^{ik^y l}) \mathcal{E} e^{-i\omega t} \\ &= (\hat{x} \cos \theta - \hat{y} \sin \theta) \mathcal{E} e^{i(k^x l - \omega t)}, \end{aligned}$$

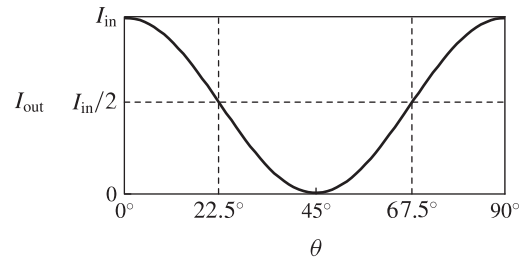
linearly polarized with partial transmission through polarizer:

$$\begin{aligned} \mathbf{E}_{out}(z = l + d, t) &= \hat{p} [(\hat{x} \cos \theta - \hat{y} \sin \theta) \cdot \hat{p}] \mathcal{E} e^{i(k^x l + kd - \omega t)} \\ &= \hat{p} \mathcal{E} \cos 2\theta e^{i(k^x l + kd - \omega t)}. \end{aligned}$$

Therefore, the output intensity, which is $I_{out} \propto |\mathbf{E}_{out}|^2$, is

$$I_{out} = I_{in} \cos^2 2\theta,$$

where I_{in} is the input intensity. The device acts a variable attenuator of linear polarized light with the output light intensity varying as a function of θ as plotted. Variation of the angle θ to vary the attenuation can be accomplished by rotating the half-wave plate while keeping the input light polarization direction and the output polarizer axis fixed and aligned to each other.



1.6.3 From the given dielectric permittivity tensor, the principal axes of this crystal are \hat{x} , \hat{y} , and \hat{z} with corresponding principal indices being $n_x = \sqrt{2.25}$, $n_y = \sqrt{2.13}$, and $n_z = \sqrt{2.02}$. The wavelength of the wave in the crystal is determined only by the polarization direction, but not the propagation direction, of the wave. The free-space wavelength is $\lambda = 600$ nm.

(a) The polarization is along \hat{x} ; thus, the wavelength is

$$\frac{\lambda}{n_x} = \frac{600 \text{ nm}}{\sqrt{2.25}} = 400 \text{ nm}.$$

(b) The polarization is along \hat{y} ; thus, the wavelength is

$$\frac{\lambda}{n_y} = \frac{600 \text{ nm}}{\sqrt{2.13}} = 411 \text{ nm}.$$

(c) The polarization is along \hat{x} ; thus, the wavelength is

$$\frac{\lambda}{n_x} = \frac{600 \text{ nm}}{\sqrt{2.25}} = 400 \text{ nm}.$$

(d) The polarization is along \hat{z} ; thus, the wavelength is

$$\frac{\lambda}{n_z} = \frac{600 \text{ nm}}{\sqrt{2.02}} = 422 \text{ nm}.$$

1.6.4 (a) Let $a\hat{x}_1 + b\hat{x}_2 + c\hat{x}_3$ be the general form of one of the new axes with $\xi = n^2$ as its corresponding eigenvalue:

$$\begin{bmatrix} 4.786 & 0 & 0.168 \\ 0 & 5.01 & 0 \\ 0.168 & 0 & 4.884 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \xi \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In order to have nontrivial solution, the following condition has to be satisfied:

$$\begin{vmatrix} 4.786 - \xi & 0 & 0.168 \\ 0 & 5.01 - \xi & 0 \\ 0.168 & 0 & 4.884 - \xi \end{vmatrix} = 0,$$

that is,

$$(4.786 - \xi)(5.01 - \xi)(4.884 - \xi) + 0.168^2(\xi - 5.01) = 0.$$

Solution of this equation yields the following eigenvalues:

$$\xi = 5.01, 5.01, 4.66.$$

The two eigenvectors, designated as \hat{x} and \hat{y} principal axes, corresponding to the degenerate eigenvalue $\xi = 5.01$ are found as

$$\begin{bmatrix} 4.786 & 0 & 0.168 \\ 0 & 5.01 & 0 \\ 0.168 & 0 & 4.884 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 5.01 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow -0.224a + 0.168c = 0,$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \hat{x} = 0.6\hat{x}_1 + 0.8\hat{x}_3, \quad \hat{y} = \hat{x}_2.$$

The third eigenvector, designated as \hat{z} principal axis, corresponding to the nondegenerate eigenvalue $\xi = 4.66$ is found as

$$\hat{z} = \hat{x} \times \hat{y} = -0.8\hat{x}_1 + 0.6\hat{x}_3.$$

Thus, we have the principal axes

$$\begin{aligned} \hat{x} &= 0.6\hat{x}_1 + 0.8\hat{x}_3, \\ \hat{y} &= \hat{x}_2, \\ \hat{z} &= -0.8\hat{x}_1 + 0.6\hat{x}_3, \end{aligned}$$

with the corresponding principal indices of refraction being

$$\begin{aligned} n_x &= (5.01)^{1/2} = 2.2383, \\ n_y &= n_x = 2.2383, \\ n_z &= (4.66)^{1/2} = 2.1587. \end{aligned}$$

(b) In the old coordinate system,

$$\eta = \begin{pmatrix} 4.786 & 0 & 0.168 \\ 0 & 5.01 & 0 \\ 0.168 & 0 & 4.884 \end{pmatrix}^{-1} = \begin{pmatrix} 4.786^{-1} & 0 & -0.00719 \\ 0 & 5.01^{-1} & 0 \\ -0.00719 & 0 & 4.884^{-1} \end{pmatrix},$$

and the index ellipsoid is expressed as

$$\frac{x_1^2}{4.786} + \frac{x_2^2}{5.01} + \frac{x_3^2}{4.884} - 0.0144x_1x_3 = 1,$$

In the new coordinate system,

$$\eta = \begin{pmatrix} 5.01 & 0 & 0 \\ 0 & 5.01 & 0 \\ 0 & 0 & 4.66 \end{pmatrix}^{-1} = \begin{pmatrix} 5.01^{-1} & 0 & 0 \\ 0 & 5.01^{-1} & 0 \\ 0 & 0 & 4.66^{-1} \end{pmatrix},$$

and the index ellipsoid is expressed as

$$\frac{x^2}{5.01} + \frac{y^2}{5.01} + \frac{z^2}{4.66} = 1$$

(c) The crystal is uniaxial with its optical axis being \hat{z} .

(d) If the wave is linearly polarized, it can be arranged (i) to propagate along \hat{z} with its polarization along any direction in the xy plane, or (ii) to propagate along any direction in the xy plane with its polarization along \hat{z} , or (iii) to propagate along any direction in the xy plane with its polarization also in the xy plane but perpendicular to the direction of propagation. If it is circularly polarized, we can only arrange the wave to propagate along \hat{z} with its circular polarization lying in the xy plane to keep the polarization unchanged.

(e) A quarter-wave plate can be made with a plate cut in a way that its surface is perpendicular to \hat{z} so that the field of a wave entering it at normal incidence has a component in z , seeing an index $n_z = 2.1587$, and another component in a plane perpendicular to z , seeing an index $n_x = n_y = 2.2383$. For $\lambda = 1 \mu\text{m}$, the thickness of the quarter-wave plate is an odd integral multiple of

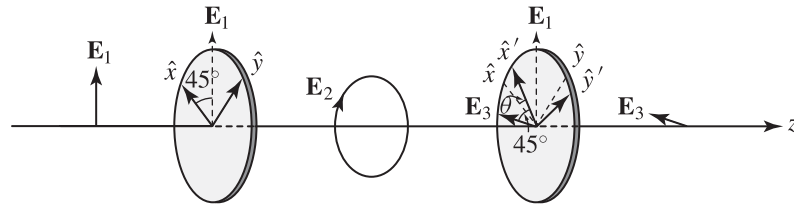
$$l_{\lambda/4} = \frac{\lambda}{4|n_x - n_z|} = \frac{1 \times 10^{-6}}{4 \times |2.2383 - 2.1587|} \text{ m} = 3.14 \mu\text{m}.$$

Any odd integral multiple of $l_{\lambda/4}$ can be the thickness.

1.6.5 If the wave is linearly polarized, the linear polarization will remain unchanged throughout with (i) propagation along any direction that is perpendicular to a principal axis with the optical field polarized along that principal axis, or (ii) propagation along an optical axis with the optical field polarized in any direction perpendicular to the optical axis, or (iii) propagation in an ordinary plane perpendicular to an optical axis with the polarization also lying in the same ordinary plane but perpendicular to both the propagation direction and the optical axis. If the wave is not linearly polarized, only propagation along an optical axis can keep the polarization unchanged throughout.

1.6.6 To answer the questions with illustration, we consider an arrangement with two quarter-wave plates in tandem as shown. Both have a thickness of $l = l_{\lambda/4}$ and are assumed to have a fast x axis and a slow y axis so that $(k^y - k^x)l = \pi/2$. The axes, \hat{x}' and \hat{y}' , of the second quarter-wave plate are rotated an angle θ with respect to those, \hat{x} and \hat{y} , of the first one so that

$$\begin{aligned} \hat{x}' &= \hat{x} \cos \theta + \hat{y} \sin \theta, & \hat{y}' &= -\hat{x} \sin \theta + \hat{y} \cos \theta, \\ \Rightarrow \hat{x} &= \hat{x}' \cos \theta - \hat{y}' \sin \theta, & \hat{y} &= \hat{x}' \sin \theta + \hat{y}' \cos \theta. \end{aligned}$$



(a) We first show that a linearly polarized field can be converted to a circularly polarized field with a quarter-wave plate. To do so, the linear polarization direction of the input field has to be in a direction that makes an angle of 45° with one of the axis of the quarter-wave plate so that the x and y components of the field have equal magnitudes. Let us take x and y components of the field to be in phase. Then, we have

linearly polarized input before the first quarter-wave plate:

$$\mathbf{E}_1 = \mathcal{E} \frac{\hat{x} + \hat{y}}{\sqrt{2}} e^{-i\omega t},$$

circularly polarized field after the first quarter-wave plate:

$$\begin{aligned}\mathbf{E}_2 &= \frac{\mathcal{E}}{\sqrt{2}} \left(\hat{x} e^{ik^x l} + \hat{y} e^{ik^y l} \right) e^{-i\omega t} \\ &= \frac{\mathcal{E}}{\sqrt{2}} (\hat{x} + i\hat{y}) e^{ik^x l - i\omega t},\end{aligned}$$

which is left-circularly polarized. A right-circularly polarized output can be obtained if the polarization direction of the input field is oriented in a way that its x and y components are equal in magnitude but 180° out of phase.

(b) We now show that a circularly polarized field can be converted to a linearly polarized field with a quarter-wave plate. Referring to the illustration above, we use the field \mathbf{E}_2 from the output of the first quarter-wave plate for the input to the second quarter-wave plate. Because the axes of the second quarter-wave plates are \hat{x}' and \hat{y}' , we first translate \mathbf{E}_2 into this new coordinate frame. Then, we have

circularly polarized field before the second quarter-wave plate:

$$\begin{aligned}\mathbf{E}_2 &= \frac{\mathcal{E}}{\sqrt{2}} (\hat{x} + i\hat{y}) e^{ik^x l - i\omega t}, \\ &= \frac{\mathcal{E}}{\sqrt{2}} [(\hat{x}' \cos \theta - \hat{y}' \sin \theta) + i(\hat{x}' \sin \theta + \hat{y}' \cos \theta)] e^{ik^x l - i\omega t}, \\ &= \frac{\mathcal{E}}{\sqrt{2}} (\hat{x}' + i\hat{y}') e^{i\theta} e^{ik^x l - i\omega t},\end{aligned}$$

linearly polarized output after the second quarter-wave plate:

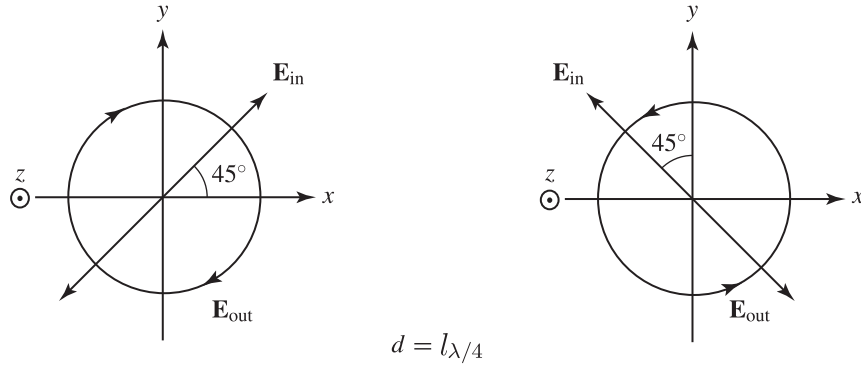
$$\begin{aligned}\mathbf{E}_3 &= \frac{\mathcal{E}}{\sqrt{2}} \left(\hat{x}' e^{ik^x l} + i\hat{y}' e^{ik^y l} \right) e^{i\theta} e^{ik^x l - i\omega t}, \\ &= \frac{\mathcal{E}}{\sqrt{2}} (\hat{x}' - \hat{y}') e^{i\theta} e^{2ik^x l - i\omega t},\end{aligned}$$

which is linearly polarized in a direction that is at 45° with respect to \hat{x}' and $-\hat{y}'$. Compared to the polarization direction of \mathbf{E}_1 , which is at 45° with respect to both \hat{x} and \hat{y} , the polarization direction of \mathbf{E}_3 is rotated by an angle of $90^\circ - \theta$. Thus, the direction of the output linear polarization can be controlled by rotating the second quarter-wave plate.

1.6.7 (a) The shortest distance is the quarter-wave length:

$$l_{\lambda/4} = \frac{\lambda}{4|n_x - n_y|} = \frac{1}{4 \times |1.55 - 1.52|} \mu\text{m} = 8.33 \mu\text{m}.$$

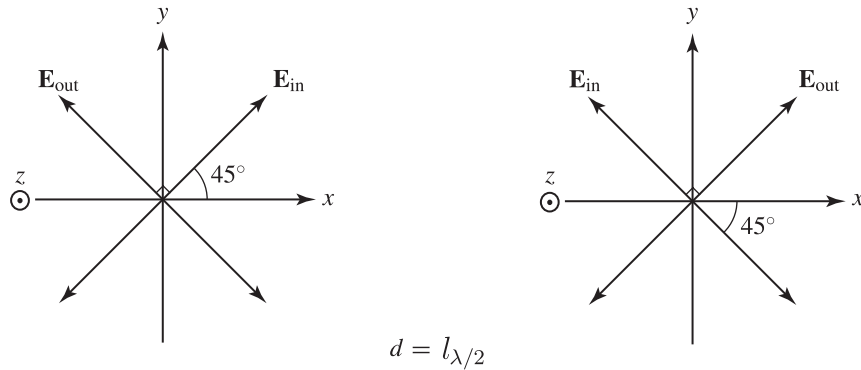
The input linear polarization must be oriented in a direction that makes 45° with respect to \hat{x} or \hat{y} on either side of the axis, as shown. The wave is circularly polarized, but at alternating sense, by traveling a distance $l = (2n + 1)l_{\lambda/4}$ that is an odd multiple of $l_{\lambda/4}$ for n being a whole number.



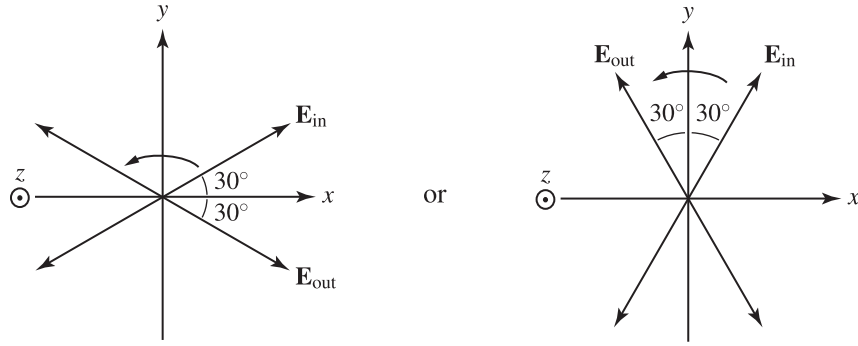
(b) The shortest distance is the half-wave length:

$$l_{\lambda/2} = \frac{\lambda}{2|n_x - n_y|} = \frac{1}{2 \times |1.55 - 1.52|} \mu\text{m} = 16.67 \mu\text{m}.$$

The input linear polarization must be oriented in a direction that makes 45° with respect to \hat{x} or \hat{y} on either side of the axis, as shown. The wave is linearly polarized, but with alternating phase of π in the field, by traveling a distance $l = (2n + 1)l_{\lambda/2}$ that is an odd multiple of $l_{\lambda/2}$ for n being a whole number.



(c) The shortest distance is the half-wave length $l_{\lambda/2} = 16.67 \mu\text{m}$ as obtained in (b), but the input linear polarization must be oriented in a direction that makes 30° with respect to \hat{x} or \hat{y} on either side of the axis, as shown. Then, the output polarization direction is rotated 60° with respect to that of the input polarization direction. The wave is linearly polarized at this direction, but with alternating phase of π in the field, by traveling a distance $l = (2n + 1)l_{\lambda/2}$ that is an odd multiple of $l_{\lambda/2}$ for n being a whole number.



1.6.8 (a) The required thickness is the quarter-wave length

$$l_{\lambda/4} = \frac{\lambda}{4(n_e - n_o)} = \frac{600 \times 10^{-9}}{4 \times (1.55332 - 1.54423)} \text{ m} = 16.5 \text{ } \mu\text{m}$$

or any of its odd multiples like 49.5 μm , 82.5 μm , etc. The quartz plate should be arranged such that its surface is normal to the propagation direction of the wave and its z axis is at 45° from the polarization direction of the input field.

(b) The required thickness is the half-wave length

$$l_{\lambda/2} = \frac{\lambda}{2(n_e - n_o)} = \frac{600 \times 10^{-9}}{2 \times (1.55332 - 1.54423)} \text{ m} = 33 \text{ } \mu\text{m}$$

or any of its odd multiples like 99 μm , 165 μm , etc. The quartz plate should be arranged such that its surface is normal to the propagation direction of the wave and its z axis is at 25° or 65° , which is $90^\circ - 25^\circ$, from the polarization direction of the input field.

(c) The required thickness is the full-wave length

$$l_\lambda = \frac{\lambda}{n_e - n_o} = \frac{600 \times 10^{-9}}{1.55332 - 1.54423} \text{ m} = 66 \text{ } \mu\text{m}$$

or any of its integral multiples like 132 μm , 198 μm , etc. The quartz plate should be arranged such that its surface is normal to the propagation direction of the wave, but its z axis can be arbitrarily oriented with respect to the polarization direction of the input field.

1.6.9 The thicknesses of quarter-wave and half-wave plates at a wavelength λ are, respectively, odd integral multiples of quarter-wave and half-wave lengths:

$$l_{\text{QW}}(\lambda) = (2n + 1)l_{\lambda/4} = \frac{(2n + 1)\lambda}{4|n_y - n_x|}, \quad l_{\text{HW}}(\lambda) = (2n + 1)l_{\lambda/2} = \frac{(2n + 1)\lambda}{2|n_y - n_x|},$$

where n can be any whole number. For a quarter-wave plate for $\lambda = 1 \text{ } \mu\text{m}$ to act as a half-wave plate for another wavelength λ' if dispersion is neglected, it is required that

$$\begin{aligned} l_{\text{QW}}(\lambda) &= l_{\text{HW}}(\lambda') \Rightarrow \frac{(2n + 1)\lambda}{4|n_y - n_x|} = \frac{(2n' + 1)\lambda'}{2|n_y - n_x|}, \\ \Rightarrow \lambda' &= \frac{2n + 1}{2(n' + 1)}\lambda = \frac{2n + 1}{2(2n' + 1)} \text{ } \mu\text{m}, \end{aligned}$$

where n is a whole number determined by the thickness of the wave plate but n' can be any whole number. In case it is chosen that $n' = n$, we have $\lambda' = 500$ nm. There are many other possibilities. For example, $\lambda' = (1/6) \mu\text{m} = 167$ nm for $n = 0$ and $n' = 1$, or $\lambda = (3/2) \mu\text{m} = 1.5 \mu\text{m}$ for $n = 1$ and $n' = 0$, etc.

For the wave to return to its original polarization state after traveling through a wave plate, the wave plate must act as a full-wave plate, which can have a thickness that is any positive integral multiple of full-wave length. For a quarter-wave plate for $\lambda = 1 \mu\text{m}$ to act as a full-wave plate for another wavelength λ' if dispersion is neglected, it is required that

$$l_{\text{QW}}(\lambda) = l_{\text{FW}}(\lambda') \Rightarrow \frac{(2n+1)\lambda}{4|n_y - n_x|} = \frac{n'\lambda'}{|n_y - n_x|},$$

$$\Rightarrow \lambda' = \frac{2n+1}{4n'}\lambda = \frac{2n+1}{4n'} \mu\text{m},$$

where n is a whole number determined by the thickness of the wave plate but n' can be any positive integer. In case it is chosen that $n = 0$ and $n' = 1$, we have $\lambda' = 250$ nm. There are many other possibilities. For example, $\lambda' = (3/4) \mu\text{m} = 750$ nm for $n = 1$ and $n' = 1$, or $\lambda = (5/4) \mu\text{m} = 1.25 \mu\text{m}$ for $n = 2$ and $n' = 1$, etc.

1.6.10 (a) The required thickness of the quartz waveplate is the half-wave length

$$l_{\lambda/2} = \frac{\lambda}{2(n_e - n_o)} = \frac{600 \times 10^{-9}}{2 \times (1.55332 - 1.54423)} \text{ m} = 33 \mu\text{m}$$

or any of its odd multiples like $99 \mu\text{m}$, $165 \mu\text{m}$, etc. The quartz plate has to be arranged such that its surface is normal to the propagation direction of the wave and its z axis is at 30° or 60° , which is $90^\circ - 30^\circ$, from the polarization direction of the input field.

(b) The thickness, l , of this half-wave plate can be any odd multiple of $l_{\lambda/2}$ found in (a) for $\lambda = 600$ nm. For the waveplate to be used as a polarization rotator for linearly polarized light at another wavelength λ' , it has to be also an odd multiple of $l_{\lambda'/2}$. Thus, if dispersion can be neglected,

$$l = (2n+1)l_{\lambda/2} = (2n'+1)l_{\lambda'/2},$$

$$\Rightarrow \frac{(2n+1)\lambda}{2(n_e - n_o)} = \frac{(2n'+1)\lambda'}{2(n_e - n_o)},$$

$$\Rightarrow \lambda' = \frac{2n+1}{2n'+1}\lambda = \frac{2n+1}{2n'+1} \times 600 \text{ nm},$$

where n is a whole number determined by the thickness of the plate but n' can be any whole number excluding $n' = n$. In the case of $n = 0$, the possible wavelengths are $\lambda' = 200$ nm for $n' = 1$, $\lambda' = 120$ nm for $n' = 2$, etc. In the case of $n = 1$, the possible wavelengths are $\lambda' = 1.8 \mu\text{m}$ for $n' = 0$, $\lambda' = 360$ nm for $n' = 2$, etc.

(c) For the waveplate to be used to convert linear polarization into circular polarization at another wavelength λ' , it has to be also an odd multiple of $l_{\lambda'/4}$. Thus, if dispersion can be neglected,

$$l = (2n+1)l_{\lambda/2} = (2n'+1)l_{\lambda'/4},$$

$$\Rightarrow \frac{(2n+1)\lambda}{2(n_e - n_o)} = \frac{(2n'+1)\lambda'}{4(n_e - n_o)},$$

$$\Rightarrow \lambda' = \frac{2n+1}{2n'+1}2\lambda = \frac{2n+1}{2n'+1} \times 1.2 \mu\text{m},$$

where n is a whole number determined by the thickness of the plate but n' can be any whole number including $n' = n$. For any plate thickness, there is always the choice of $n' = n$ for $\lambda' = 1.2 \mu\text{m}$. In the case of $n = 0$, other possible wavelengths are $\lambda' = 400 \text{ nm}$ for $n' = 1$, $\lambda' = 240 \text{ nm}$ for $n' = 2$, etc. In the case of $n = 1$, other possible wavelengths are $\lambda' = 3.6 \mu\text{m}$ for $n' = 0$, $\lambda' = 720 \text{ nm}$ for $n' = 2$, etc.

(d) For this purpose, the thickness, which has to be an odd multiple of $l_{\lambda/2}$, has to be

$$\begin{aligned} 1 \text{ mm} < l &= (2n + 1)l_{\lambda/2} = 33(2n + 1) \mu\text{m} < 1.5 \text{ mm}, \\ \Rightarrow 15 &\leq n \leq 22, \\ \Rightarrow l &= 1.023 \text{ mm}, 1.089 \text{ mm}, 1.155 \text{ mm}, 1.221 \text{ mm}, \\ &1.287 \text{ mm}, 1.353 \text{ mm}, 1.419 \text{ mm}, 1.485 \text{ mm}. \end{aligned}$$

Any of the eight possible thicknesses can be chosen.

- 1.6.11 (a) At $\lambda = 1 \mu\text{m}$, $n_o = 2.486$ and $n_e = 2.750$ from (1.181) and (1.182), respectively. The thickness of a first-order half-wave plate is

$$l = l_{\lambda/2} = \frac{\lambda}{2(n_e - n_o)} = \frac{1}{2 \times (2.750 - 2.486)} \mu\text{m} = 1.89 \mu\text{m}.$$

The plate should be arranged such that its uniaxial axis is 30° or 60° , which is $90^\circ - 30^\circ$, with respect to the polarization direction of the incident beam.

(b) Note that the dispersion can be considered using (1.181) and (1.182). For the plate to function as a half-wave plate at another wavelength λ' , l must be an odd integral multiple of $l_{\lambda'/2}$:

$$\begin{aligned} l &= (2n + 1)l_{\lambda'/2} = \frac{(2n + 1)\lambda'}{2[n_e(\lambda') - n_o(\lambda')]}, \\ \Rightarrow \lambda' &= \frac{2[n_e(\lambda') - n_o(\lambda')]}{2n + 1} l = \frac{2[n_e(\lambda') - n_o(\lambda')]}{2n + 1} \times 1.89 \mu\text{m}. \end{aligned}$$

For $n = 0$, we already have $\lambda' = \lambda = 1 \mu\text{m}$ given initially. To find another wavelength, we solve this relation by taking $n = 1$ and using (1.181) and (1.182) for n_o and n_e to find that $\lambda' = 440 \text{ nm}$. For the plate to function as a quarter-wave plate at a wavelength λ'' , l must be an odd integral multiple of $l_{\lambda''/4}$:

$$\begin{aligned} l &= (2n + 1)l_{\lambda''/4} = \frac{(2n + 1)\lambda''}{4[n_e(\lambda'') - n_o(\lambda'')]}, \\ \Rightarrow \lambda'' &= \frac{4[n_e(\lambda'') - n_o(\lambda'')]}{2n + 1} l = \frac{4[n_e(\lambda'') - n_o(\lambda'')]}{2n + 1} \times 1.89 \mu\text{m}. \end{aligned}$$

We solve this relation by using (1.181) and (1.182) for n_o and n_e to find that $\lambda'' = 1.922 \mu\text{m}$ for $n = 0$, $\lambda'' = 703 \text{ nm}$ for $n = 1$, and $\lambda'' = 488 \text{ nm}$ for $n = 2$.

- 1.6.12 (a) From (1.122), $\hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$. Then, using (1.121), we find that

$$\begin{aligned} \hat{e}_o &= \frac{1}{\sin \theta} \hat{k} \times \hat{z} \\ &= \frac{1}{\sin \theta} (\hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta) \times \hat{z} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sin \theta} (-\hat{y} \sin \theta \cos \phi + \hat{x} \sin \theta \sin \phi) \\
 &= \hat{x} \sin \phi - \hat{y} \cos \phi \quad \Rightarrow \quad (1.123), \\
 \hat{e}_e &= \hat{e}_o \times \hat{k} \\
 &= (\hat{x} \sin \phi - \hat{y} \cos \phi) \times (\hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta) \\
 &= \hat{z} \sin \theta \cos^2 \phi + \hat{z} \sin \theta \sin^2 \phi - \hat{y} \cos \theta \sin \phi - \hat{x} \cos \theta \cos \phi \\
 &= -\hat{x} \cos \theta \cos \phi - \hat{y} \cos \theta \sin \phi + \hat{z} \sin \theta \quad \Rightarrow \quad (1.124).
 \end{aligned}$$

(b) For a uniaxial crystal, $n_x = n_y = n_o$ and $n_z = n_e$. From (1.117), the index ellipsoid is described by the following equation:

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \quad \Rightarrow \quad \frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1.$$

The index $n_e(\theta)$ is determined by the intersection of the vector \hat{e}_e with the index ellipsoid, as shown in Fig. 1.11. In Fig. 1.11, vector \hat{k} is on the yz plane so that $\phi = 90^\circ$. For a general case, consider $\hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$. Then, the intersection of \hat{e}_e with the index ellipsoid has the coordinates $x = -n_e(\theta) \cos \theta \cos \phi$, $y = -n_e(\theta) \cos \theta \sin \phi$, and $z = n_e(\theta) \sin \theta$. Thus,

$$\begin{aligned}
 &\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1, \\
 \Rightarrow &\frac{n_e^2(\theta) \cos^2 \theta \cos^2 \phi}{n_o^2} + \frac{n_e^2(\theta) \cos^2 \theta \sin^2 \phi}{n_o^2} + \frac{n_e^2(\theta) \sin^2 \theta}{n_e^2} = 1, \\
 \Rightarrow &\frac{n_e^2(\theta) \cos^2 \theta}{n_o^2} + \frac{n_e^2(\theta) \sin^2 \theta}{n_e^2} = 1, \\
 \Rightarrow &\frac{1}{n_e^2(\theta)} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2} \quad \Rightarrow \quad (1.125).
 \end{aligned}$$

1.6.13 The propagation constant of an optical wave is determined by the normal mode polarization, which is perpendicular to the propagation direction \hat{k} . In a birefringent crystal, the normal modes are determined by the \mathbf{D} field components, because they are all perpendicular to \hat{k} , but are not determined by the \mathbf{E} field, because \mathbf{E} is not necessarily perpendicular to \hat{k} . However, the components of \mathbf{D} projected on the principal axes are related to those of \mathbf{E} through (1.109): $D_x = \epsilon_x E_x = \epsilon_0 n_x^2 E_x$, $D_y = \epsilon_y E_y = \epsilon_0 n_y^2 E_y$, and $D_z = \epsilon_z E_z = \epsilon_0 n_z^2 E_z$. From these discussions, we can understand the difference between (1.118) and (1.126).

In the case of (1.118), the wave propagates along principal axis \hat{z} so that $\hat{k} = \hat{z}$. In this special situation, both $\mathbf{D} \perp \hat{k}$ and $\mathbf{E} \perp \mathbf{k}$. Thus, both \mathbf{D} and \mathbf{E} can be decomposed into two normal-mode components along the two principal axes \hat{x} and \hat{y} , which propagate with wavevectors $\mathbf{k}^x = k^x \hat{k} = k^x \hat{z}$ and $\mathbf{k}^y = k^y \hat{k} = k^y \hat{z}$, respectively. Therefore, \mathbf{D} and \mathbf{E} can be respectively expressed as

$$\begin{aligned}
 \mathbf{D} &= \hat{x} D_x e^{ik^x z - i\omega t} + \hat{y} D_y e^{ik^y z - i\omega t} \\
 &= \hat{x} \epsilon_x \mathcal{E}_x e^{ik^x z - i\omega t} + \hat{y} \epsilon_y \mathcal{E}_y e^{ik^y z - i\omega t}, \\
 &= \hat{x} n_x^2 \epsilon_0 \mathcal{E}_x e^{ik^x z - i\omega t} + \hat{y} n_y^2 \epsilon_0 \mathcal{E}_y e^{ik^y z - i\omega t}, \\
 \mathbf{E} &= \hat{x} \mathcal{E}_x e^{ik^x z - i\omega t} + \hat{y} \mathcal{E}_y e^{ik^y z - i\omega t} \quad \Rightarrow \quad (1.118).
 \end{aligned}$$

Note that though both \mathbf{D} and \mathbf{E} can be expressed in terms of normal modes as above, $\mathbf{D} \nparallel \mathbf{E}$ in general if $\epsilon_x \neq \epsilon_y$, i.e., if $n_x \neq n_y$.

In the case of (1.126), the wave propagates in a general direction not along a principal axis so that \hat{k} does not line up with any principal axis. In this situation, $\mathbf{E} \nparallel \hat{k}$ though $\mathbf{D} \perp \hat{k}$. Therefore, the normal modes are defined by the ordinary and extraordinary components of \mathbf{D} but not by the components of \mathbf{E} . Thus, \mathbf{D} can be expressed in terms of normal modes in (1.126), but \mathbf{E} cannot be expressed in a similar form. Because $\mathbf{D}_o \parallel \mathbf{E}_o$, we have $D_o = n_o^2 \epsilon_0 E_o$. However, because $\mathbf{D}_e \nparallel \mathbf{E}_e$, there does not exist such a simple relation between D_e and E_e . In the case of a uniaxial crystal with \hat{z} being the optical axis, they are related as $D_x^e = n_o^2 \epsilon_0 E_x^e$, $D_y^e = n_o^2 \epsilon_0 E_y^e$, and $D_z^e = n_e^2 \epsilon_0 E_z^e$. Thus,

$$\begin{aligned} \mathbf{D} &= \hat{e}_o \mathcal{D}_o e^{ik_o \hat{k} \cdot \mathbf{r} - i\omega t} + \hat{e}_e \mathcal{D}_e e^{ik_e \hat{k} \cdot \mathbf{r} - i\omega t} \Rightarrow (1.126), \\ &= \hat{e}_o n_o^2 \epsilon_0 \mathcal{E}_o e^{ik_o \hat{k} \cdot \mathbf{r} - i\omega t} + (\hat{x} n_o^2 \epsilon_0 \mathcal{E}_x^e + \hat{y} n_o^2 \epsilon_0 \mathcal{E}_y^e + \hat{z} n_e^2 \epsilon_0 \mathcal{E}_z^e) e^{ik_e \hat{k} \cdot \mathbf{r} - i\omega t}, \\ \mathbf{E} &= \hat{e}_o \mathcal{E}_o e^{ik_o \hat{k} \cdot \mathbf{r} - i\omega t} + (\hat{x} \mathcal{E}_x^e + \hat{y} \mathcal{E}_y^e + \hat{z} \mathcal{E}_z^e) e^{ik_e \hat{k} \cdot \mathbf{r} - i\omega t}, \\ &= \mathcal{E}_o e^{ik_o \hat{k} \cdot \mathbf{r} - i\omega t} + \mathcal{E}_e e^{ik_e \hat{k} \cdot \mathbf{r} - i\omega t}. \end{aligned}$$

We see that while \mathbf{E} still consists of two components, \mathbf{E}_o , which has an amplitude \mathcal{E}_o and a propagation constant k_o , and \mathbf{E}_e , which has an amplitude \mathcal{E}_e and a propagation constant k_e , the extraordinary component cannot be expressed as $\mathcal{E}_e = \hat{e}_e \mathcal{E}_e$ because $\mathcal{E}_e \nparallel \hat{e}_e$ in general.

1.6.14 Because $n_x = n_y = n_o$ and $n_z = n_e$, we have $D_z^e = n_e^2 \epsilon_0 E_z^e$ and $D_{xy}^e = n_o^2 \epsilon_0 E_{xy}^e$. Referring to Fig. 1.13, we find that

$$\begin{aligned} D_z^e &= n_e^2 \epsilon_0 E_z^e, \quad D_{xy}^e = n_o^2 \epsilon_0 E_{xy}^e, \\ \Rightarrow D_e \sin \theta &= n_e^2 \epsilon_0 E_e \sin \psi_e, \quad D_e \cos \theta = n_o^2 \epsilon_0 E_e \cos \psi_e, \\ \Rightarrow \tan \theta &= \frac{n_e^2}{n_o^2} \tan \psi_e, \\ \Rightarrow \psi_e &= \tan^{-1} \left(\frac{n_o^2}{n_e^2} \tan \theta \right), \\ \Rightarrow \alpha &= \psi_e - \theta = \tan^{-1} \left(\frac{n_o^2}{n_e^2} \tan \theta \right) - \theta \Rightarrow (1.131). \end{aligned}$$

The angle θ for the largest walk-off is found by taking $d\alpha/d\theta = 0$:

$$\begin{aligned} \frac{d\alpha}{d\theta} &= \frac{1}{1 + (n_o^4/n_e^4) \tan^2 \theta} \frac{n_o^2}{n_e^2} \sec^2 \theta - 1 = 0, \\ \Rightarrow \frac{n_o^2}{n_e^2} \sec^2 \theta &= 1 + \frac{n_o^4}{n_e^4} \tan^2 \theta, \\ \Rightarrow \frac{n_o^2}{n_e^2} + \frac{n_o^2}{n_e^2} \tan^2 \theta &= 1 + \frac{n_o^4}{n_e^4} \tan^2 \theta, \\ \Rightarrow \frac{n_o^2}{n_e^2} \tan^2 \theta &= 1, \\ \Rightarrow \theta &= \tan^{-1} \frac{n_e}{n_o}. \end{aligned}$$

This propagation angle gives the largest walk-off.