Detailed Solutions to Exercises

These are my solutions to the exercises from "A Basis Theory Primer." Of course, many problems have solutions other than the ones I sketch. Please send comments and corrections to "`heil@math.gatech.edu`".

Detailed Solutions to Exercises from Chapter 1

1.1 Since $\|\cdot\|$ is a norm, we must have $\lambda = \|1\| \neq 0$. Then given any $x \in \mathbf{F}$, we have $\|x\| = \|x \cdot 1\| = |x|\,\|1\| = \lambda\,|x|$.

1.2 (a) Suppose $x_n \to x$, and choose $\varepsilon > 0$. Then there exists $N > 0$ such that $\|x - x_n\| < \varepsilon$ for all $n > N$. Hence, by the Triangle Inequality, if $m, n > N$ then $\|x_m - x_n\| \le \|x_m - x\| + \|x - x_n\| < 2\varepsilon$. Thus $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy.

(b) Suppose that $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy. Then there exists an $N > 0$ such that $\|x_m - x_n\| < 1$ for all $m, n \ge N$. Therefore, for $n \ge N$ we have

$$\|x_n\| \;=\; \|x_n - x_N + x_N\| \;\le\; \|x_n - x\| + \|x_N\| \;\le\; 1 + \|x_N\|.$$

Hence for any n we have

$$\|x_n\| \;\le\; \max\bigl\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\bigr\},$$

so $\{x_n\}$ is bounded.

(c) Given $x, y \in H$, we have

$$\|x\| \;=\; \|(x - y) + y\| \;\le\; \|x - y\| + \|y\|,$$

so $\|x\| - \|y\| \le \|x - y\|$. Reversing the roles of x and y, we obtain $\|y\| - \|x\| \le \|x - y\|$, so we have $\bigl|\|x\| - \|y\|\bigr| \le \|x - y\|$.

(d) By the Reverse Triangle Inequality, if $x_n \to y$, then

$$\Big| \|x\| - \|x_n\| \Big| \;\le\; \|x - x_n\| \;\to\; 0.$$

(e) If $x_n \to x$ and $y_n \to y$, then

$$\|(x+y) - (x_n + y_n)\| \;=\; \|(x - x_n) + (y - y_n)\| \;\le\; \|x - x_n\| + \|y - y_n\| \;\to\; 0.$$

(f) Suppose $x_n \to x$ and $c_n \to c$. Then $C = \sup |c_n| < \infty$, so

$$\begin{aligned}\|cx - c_n x_n\| &\le \|cx - c_n x\| + \|c_n x - c_n x_n\| \\ &= |c - c_n|\,\|x\| + |c_n|\,\|x - x_n\| \\ &\le |c - c_n|\,\|x\| + C\,\|x - x_n\| \;\to\; 0.\end{aligned}$$

1.3 (a) Suppose first that $1 \le p < \infty$ and $q = \infty$. Given $x = (x_1, \dots, x_d) \in \mathbf{F}^d$, we have $|x_k| \le \|x\|_\infty$ for each k. Therefore

$$|x|_p \;=\; \big(|x_1|^p + \cdots + |x_d|^p\big)^{1/p} \;\le\; \big(d\,\|x\|_\infty\big)^{1/p} \;=\; d^{1/p}\,\|x\|_\infty.$$

Conversely, $\|x\|_\infty = |x_k|$ for some particular k, so we have

$$\|x\|_\infty \;=\; |x_k| \;\le\; \big(|x_1|^p + \cdots + |x_d|^p\big)^{1/p} \;=\; \|x\|_p.$$

Hence $|\cdot|_p$ and $|\cdot|_\infty$ are equivalent norms on $\mathbf{F}^d$.

If we now choose any $1 \le p, q \le \infty$, then $|\cdot|_p$ is equivalent to $|\cdot|_\infty$, and $|\cdot|_\infty$ is equivalent to $|\cdot|_q$, so it follows that $|\cdot|_p$ is equivalent to $|\cdot|_q$.

1.4 If $\|x\| = 0$ then $c_k(x) = 0$ for each k, so $x = 0$. All of the other properties of a norm follow easily. To show completeness, note that for each $1 \le k \le d$ we have $|c_k(x)| \le \|x\|$. Also, the c_k are linear, so if $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in X, then for each fixed k we have

$$|c_k(x_m) - c_k(x_n)| \;=\; |c_k(x_m - x_n)| \;\le\; \|x_m - x_n\|.$$

This implies that $\{c_k(x_n)\}_{n\in\mathbf{N}}$ is a Cauchy sequence of scalars and hence converges to some scalar c_k. Define $x = \sum_{k=1}^{d} c_k x_k$. The fact that $x_n \to x$ then follows just as in the proof that ℓ^p is complete.

1.5 We are given that $\|x_{n+1} - x_n\| < 2^{-n}$ for every n. Choose any $\varepsilon > 0$, and let N be large enough that $2^{-N+1} < \varepsilon$. If $n > m > N$, then we have

$$\|x_n - x_m\| \;\le\; \sum_{k=m}^{n-1} \|x_{k+1} - x_k\| \;\le\; \sum_{k=m}^{\infty} \frac{1}{2^k} \;=\; \frac{1}{2^{m-1}} \;<\; \frac{1}{2^{N-1}} \;<\; \varepsilon.$$

Hence $\{x_n\}$ is Cauchy.

1.6 Suppose that every subsequence of $\{x_n\}$ has a subsequence that converges to x, but the full sequence $\{x_n\}$ does not converge to x. Then there exists an $\varepsilon > 0$ such that given any N we can find an $n > N$ such that $\|x - x_n\| > \varepsilon$.

Then we can find a subsequence $\{x_{n_k}\}$ such that $\|x - x_{n_k}\| > \varepsilon$ for every k. But then no subsequence of $\{x_{n_k}\}$ can converge to x, which is a contradiction.

1.7 It is clear that $\|\cdot\|_{\mathbf{R}}$ is a norm on $X_{\mathbf{R}}$, so the issue is to show that $X_{\mathbf{R}}$ is complete. Suppose that $\{x_n\}$ is Cauchy in $X_{\mathbf{R}}$. Since $\|x_m - x_n\| = \|x_m - x_n\|_{\mathbf{R}}$, we have that $\{x_n\}$ is Cauchy in X and therefore converges to some $x \in X$. But then $\|x - x_n\|_{\mathbf{R}} = \|x - x_n\| \to 0$, so x_n converges to x in $X_{\mathbf{R}}$. Hence $X_{\mathbf{R}}$ is complete.

1.8 (a) The Triangle Inequality follows from $\mathrm{d}(f,h) = \|(f-g)+(g-h)\| \le \|f-g\| + \|g-h\| = \mathrm{d}(f,g) + \mathrm{d}(g,h)$.

(b) A sequence $\{f_n\}_{n\in\mathbf{N}}$ *converges* to $f \in X$ if $\lim_{n\to\infty} \mathrm{d}(f_n, f) = 0$, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, n \ge N, \quad \mathrm{d}(f_n, f) < \varepsilon.$$

A sequence $\{f_n\}_{n\in\mathbf{N}}$ is *Cauchy* if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, m, n \ge N, \quad \mathrm{d}(f_m, f_n) < \varepsilon.$$

(c) For each n, let x_n be a rational number such that $\pi < x_n < \pi + 1/n$. Then $\{x_n\}$ is Cauchy, but it does not converge *in the space* $\mathbf{Q}$. It does converge in the larger space $\mathbf{R}$, but since the limit does not belong to $\mathbf{Q}$, it is not convergent in $\mathbf{Q}$.

1.9 (a) Let δ_1, δ_2 denote the first two standard basis vectors. These belong to ℓ^p, but we have

$$\|x+y\|_p \;=\; (1+1)^{1/p} \;=\; 2^{1/p}$$

while

$$\|x\|_p + \|y\|_p \;=\; 1+1 \;=\; 2.$$

Since $p < 1$ we have $2^{1/p} > 2$, so the Triangle Inequality is not satisfied by $\|\cdot\|_p$.

(b) Suppose that $0 < p < 1$. Let $f(t) = (1+t)^p$ and $g(t) = 1 + t^p$ for $t > 0$. Then $f(0) = 1 = g(0)$. Also,

$$f'(t) \;=\; p\,(1+t)^{p-1} \;=\; p\,\frac{1}{(1+t)^{1-p}} \qquad \text{and} \qquad g'(t) \;=\; pt^{p-1} \;=\; p\,\frac{1}{t^{1-p}}.$$

Since $0 < 1-p < 1$, we have $t^{1-p} < (1+t)^{1-p}$, and therefore $f'(t) \le g'(t)$ for $t > 0$. Hence g is increasing faster than f, and therefore $f(t) \le g(t)$ for all $t \ge 0$. Next, given any a, $b \ge 0$, we have

$$(a+b)^p \;=\; a^p\left(1+\frac{b}{a}\right)^p \;\le\; a^p\left(1+\left(\frac{b}{a}\right)^p\right) \;=\; a^p + b^p.$$

Hence, if x, $y \in \ell^p(I)$, then

$$\|x+y\|_p^p \;=\; \sum_{k\in I} |x_k+y_k|^p \;\le\; \sum_{k\in I} \bigl(|x_k|^p+|y_k|^p\bigr) \;=\; \|x\|_p^p+\|y\|_p^p.$$

This establishes the Triangle Inequality.

(c) To show that the unit ball is not convex, note that the the standard basis vectors δ_1 and δ_2 both belong to the closed unit ball

$$D = \bigl\{x \in \ell^p : \|x\|_p \le 1\bigr\},$$

but

$$\Bigl\|\frac{\delta_1+\delta_2}{2}\Bigr\|_p^p \;=\; \Bigl(\frac{1}{2}\Bigr)^p + \Bigl(\frac{1}{2}\Bigr)^p \;=\; \frac{2}{2^p} \;=\; 2^{1-p} \;>\; 1,$$

so $(\delta_1+\delta_2)/2$ does not belong to the closed unit ball. Hence this set is not convex in ℓ^p. This also shows that if $\varepsilon > 0$ is small, then the open unit ball $B_{1+\varepsilon}(0)$ is not convex. By rescaling, the unit ball $B_1(0)$ is not convex either.

1.10 (a) Set $f(t) = t^\theta - \theta t - (1-\theta)$. Then $f'(t) = \theta t^{\theta-1} - \theta$. We have $f'(t) = 0$ if and only if $t = 1$. Also, f is increasing for $0 < t < 1$ and decreasing for $t > 1$, and $f(1) = 0$, so $f(t) \le 0$ for all $t > 0$, with equality only for $t = 1$.

(b) Note that

$$\frac{1}{p}+\frac{1}{p'} \;=\; 1, \qquad p' \;=\; \frac{p}{p-1}, \qquad \frac{p'}{p} \;=\; \frac{1}{p-1}, \qquad p' - \frac{p'}{p} \;=\; 1.$$

With $t = a^p\, b^{-p'}$ and $\theta = 1/p$, we have by part (a) that

$$a\, b^{-p'/p} \;=\; \bigl(a^p\, b^{-p'}\bigr)^{1/p} \;\le\; a^p\, b^{-p'}\,\frac{1}{p} + \Bigl(1-\frac{1}{p}\Bigr) \;=\; \frac{a^p\, b^{-p'}}{p} + \frac{1}{p'}.$$

Multiplying through by $b^{p'}$ and using the fact that $p' - (p'/p) = 1$, we obtain

$$ab \;=\; a\, b^{p'-p'/p} \;\le\; \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Equality holds if and only if $a^p b^{-p'} = 1$. This is equivalent to $b^{p'} = a^p$, or

$$b \;=\; a^{p/p'} \;=\; a^{p-1}.$$

1.11 *Case* $1 < p < \infty$. By Exercise 1.10, equality holds in $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$ if and only if $b = a^{p-1}$. For the normalized case $\|x\|_p = \|y\|_{p'} = 1$, equality in Hölder's Inequality requires that we have equality in equation (1.5), and this will happen if and only if $|y_k| = |x_k|^{p-1}$ for each k. This is equivalent to

$$|y_k|^{p'} \;=\; |y_k|^{p/(p-1)} \;=\; |x_k|^p.$$

For the nonnormalized case, if $x,\, y \ne 0$, equality holds in Hölder's Inequality if and only if it holds when we replace x and y by $x/\|x\|_p$ and $y/\|y\|_{p'}$. Therefore, we must have

$$\frac{|y_k|^{p'}}{\|y\|_{p'}^{p'}} = \left(\frac{|y_k|}{\|y\|_{p'}}\right)^{p'} = \left(\frac{|x_k|}{\|x\|_p}\right)^{p} = \frac{|x_k|^p}{\|x\|_p^p}, \qquad k \in I.$$

Hence $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ with $\alpha = \|y\|_{p'}^{p'}$ and $\beta = \|x\|_p^p$. On the other hand, if either $x = 0$ or $y = 0$, then we have equality in Hölder's Inequality, and we also have $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ with α, β not both zero.

For the converse direction, suppose that $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ for each $k \in I$, where $\alpha, \beta \in \mathbf{F}$ are not both zero. If $\alpha = 0$, then $y_k = 0$ for every k, and hence we trivially have $\|xy\|_1 = 0 = \|x\|_p\,\|y\|_{p'}$. Likewise, equality holds trivially if $\beta = 0$. Therefore, we can assume both $\alpha, \beta \neq 0$, and by dividing both sides by β, we may assume that $\beta = 1$ and $\alpha > 0$. Then we have $|y_k|^{p'} = \alpha\,|x_k|^p$, so

$$\|y\|_{p'}^{p'} = \sum_{k\in I} |y_k|^{p'} = \alpha \sum_{k\in I} |x_k|^p = \alpha\,\|x\|_p^p.$$

If either $x = 0$ or $y = 0$ then equality holds trivially in Hölder's Inequality, so let us assume both $x, y \neq 0$. Then we have

$$\frac{|y_k|^{p'}}{\|y\|_{p'}^{p'}} = \frac{\alpha |x_k|^p}{\alpha \|x\|_p^p} = \frac{|x_k|^p}{\|x\|_p^p}.$$

By the work above, this implies that equality holds in Hölder's Inequality.

Case $p = 1$, $p' = \infty$. Set $M = \sup_k |y_k|$. Suppose equality holds in Hölder's Inequality, i.e.,

$$\sum_{k\in I} |x_k y_k| = \left(\sum_{k\in I} |x_k|\right)\left(\sup_k |y_k|\right).$$

Then

$$\sum_{k\in I} |x_k y_k| = \sum_{k\in I} M\,|x_k|.$$

Hence

$$\sum_{k\in I} (M - |y_k|)\,|x_k| = 0,$$

but $0 \le M - |y_k|$ for every k, so we must have $(M - |y_k|)\,|x_k| = 0$ for every k. Thus whenever $x_k \neq 0$, we must have $|y_k| = M$.

Conversely, if $|y_k| = M$ for all k such that $x_k \neq 0$, equality holds in Hölder's Inequality.

1.12 We have $\ell^p \subseteq \ell^\infty$ for every p. Further, the constant sequence $x = (1, 1, 1, \dots)$ belongs to ℓ^∞ but not to any ℓ^p with p finite, so the inclusion is proper.

Suppose $0 < p \le q < \infty$ and $x \in \ell^p$. If $\|x\|_\infty = 1$, then

$$\|x\|_q \;=\; \biggl(\sum_{k=1}^{\infty} |x_k|^q\biggr)^{1/q} \;=\; \biggl(\sum_{k=1}^{\infty} |x_k|^p\,|x_k|^{q-p}\biggr)^{1/q}$$
$$\le\; \biggl(\sum_{k=1}^{\infty} |x_k|^p\biggr)^{1/q} \;=\; \|x\|_p^{p/q} \;\le\; \|x\|_p,$$

the last inequality following from the fact that $p/q \le 1$ and $\|x\|_p \ge \|x\|_\infty = 1$. For the general case, apply this inequality to $x/\|x\|_\infty$.

To show that the inclusion is strict, set $x_k = k^{-1/p}$. Then since $q/p > 1$, we have

$$\|x\|_q^q \;=\; \sum_{k=1}^{\infty} \frac{1}{k^{q/p}} \;<\; \infty,$$

while

$$\|x\|_p^p \;=\; \sum_{k=1}^{\infty} \frac{1}{k} \;=\; \infty.$$

Another example is $x_k = (k \log^2 k)^{-1/q}$ for $k \ge 2$. The Integral Test shows that

$$\|x\|_q^q \;=\; \sum_{k=2}^{\infty} \frac{1}{k \log^2 k} \;<\; \infty,$$

while

$$\|x\|_p^p \;=\; \sum_{k=2}^{\infty} \frac{1}{(k \log^2 k)^{p/q}} \;=\; \infty.$$

1.13 We need the following lemma.

Lemma. If $E \subseteq \mathbf{R}$ is measurable and $0 < |E| < \infty$, then there exists a measurable $F \subseteq E$ such that $|F| = |E|/2$.

Proof. Let $E_t = E \cap (-\infty, t]$. Then the sets E_t are nested increasing with t, their union is E, and their intersection is empty. Applying continuity from both above and below, which is applicable since $|E| < \infty$, we conclude that

$$\lim_{t\to\infty} |E_t| = |E| \qquad \text{and} \qquad \lim_{t\to-\infty} |E_t| = 0.$$

A similar argument shows that $|E_t|$ is a continuous function of t. Therefore there must be some t such that $|E_t| = |E|/2$. □

Now we return to the proof of the exercise. Assume that $1 \le p < q < \infty$. Taking $E_0 = E$, by applying the Lemma, we can find a set $E_1 \subseteq E$ such that $|E_1| = |E|/2$. Noting that $|E\backslash E_1| = |E|/2$, we apply the lemma to find $E_2 \subseteq E\backslash E_1$ with $|E_2| = |E|/4$, and note that E_2 is disjoint from E_1. Continuing in this way we construct disjoint $E_n \subseteq E$ such that $|E_n| = 2^{-n}\,|E|$. Consider

$$f = \sum_n 2^{n/q} \chi_{E_n}.$$

We have

$$\|f\|_{L^q}^q = \int_E |f|^q = \sum_n 2^n |E_n| = \sum_n 2^n 2^{-n} |E| = \infty,$$

so $f \notin L^q(E)$. On the other hand,

$$\begin{aligned} \|f\|_{L^p}^p = \int_E |f|^p &= \sum_n 2^{np/q} |E_n| \\ &= \sum_n 2^{np/q} 2^{-n} |E| \\ &\le \sum_n 2^{n(\frac{p}{q}-1)} |E| < \infty, \end{aligned}$$

since $p/q < 1$. Hence $f \in L^p(E)$. Note that this f is unbounded, so this is also an example of a function in $L^p(E)$ that does not belong to $L^\infty(E)$.

1.14 Suppose that $x \in \ell^q(I)$ for some finite q. Since only countably many components of x can be nonzero, it suffices to consider $I = \mathbf{N}$.

If $x = 0$ then $\|x\|_p = 0$ for every p, so we are done. Therefore, we may assume $x \neq 0$, which implies $\|x\|_\infty \neq 0$. By dividing through by $\|x\|_\infty$, we may assume that $\|x\|_\infty = 1$. Then for every p we have $1 = \|x\|_\infty \le \|x\|_p$. In particular, $|x_k| \le 1$ for every k. Therefore, for $p \ge q$ we have $|x_k|^p \le |x_k|^q$. Hence $x \in \ell^p$. Further, for $p \ge q$,

$$\begin{aligned} \|x\|_\infty = 1 \le \|x\|_p &= \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p} \\ &\le \left(\sum_{k=1}^\infty |x_k|^q\right)^{1/p} \\ &= \|x\|_q^{q/p} \\ &\to 1 = \|x\|_\infty \quad \text{as } p \to \infty, \end{aligned}$$

where the limit exists because $\|x\|_q$ is finite and nonzero.

On the other hand, the vector $x = (1, 1, 1, \dots)$ satisfies $\|x\|_\infty = 1$, but $\|x\|_p = \infty$ for every $p < \infty$.

1.15 The proof is very similar to the proof that $\ell^p(\mathbf{N})$ is a Banach space. For example, if $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $\ell^p(I)$ and we write $x_n = (x_n(i))_{i\in I}$, then for each fixed i we have that $(x_n(i))_{n\in\mathbf{N}}$ is a Cauchy sequence of scalars, and hence converges to some scalar $x(i)$. For a given n, at most countable many components of x_n can be nonzero. As a countable union of countable sets is countable, at most countably many components of x can be nonzero. An argument similar to the one used in the proof of Theorem 1.14 then shows that $x_n \to x$ in the norm of $\ell^p(I)$, so $\ell^p(I)$ is complete.

1.16 (a) The Triangle Inequality follows from Hölder's Inequality, and an argument similar to the one used in the proof of Exercise 1.3 shows that the norm $\|(x,y)\|_p$ is equivalent to the norm $\|(x,y)\|_\infty$.

(b) Suppose that X and Y are complete. If $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$, then $\{x_n\}$ is a Cauchy sequence in X since we always have $\|x\|_X \le \|(x,y)\|_p$. Hence $x_n \to x$ for some $x \in X$, and similarly $y_n \to y$ for some $y \in Y$. It then follows that $(x_n, y_n) \to (x, y)$ with respect to the norm on $X \times Y$.

1.17 (a) $\Rightarrow$. Suppose that E is closed. If $x \notin E$ then since $U = X\backslash E$ is open, there exists some $r > 0$ such that $B_r(x) \subseteq X\backslash E$. Consequently, every element of E is at least a distance r from x, and therefore x cannot be a limit point of E. Hence every limit point of E belongs to E.

$\Leftarrow$. Suppose that E is not closed. Then $X\backslash E$ is not open, so there exists some $x \notin E$ such that $B_r(x)$ is not contained in $X\backslash E$ for any $r > 0$. Therefore, for each $r = 1/n$ we can find an $x_n \in E \cap B_{1/n}(x)$. But then $x_n \to x$ and $x_n \in E$. Since $x \notin E$ while $x_n \in E$, we must have $x_n \ne x$. Therefore x is a limit point of E, so E does not contain all of its limit points.

(b) Let $F = E \cup \{x \in X : x \text{ is a limit point of } E\}$. Our goal is to show that $F = \overline{E}$.

Suppose that $x \in F^{\mathrm{C}} = X\backslash F$. Then, by definition, $x \notin E$ and x is not a limit point of E. If every open ball $B_r(x)$ contained an element of E (which necessarily must not be x), then x would be a limit point of E. Therefore, there exist some $B_r(x)$ that contains no points of E. Suppose that $B_r(x)$ contained some limit point y of E. Then there would be points $x_n \in E$ such that $x_n \to y$. But then for n large enough we would have $x_n \in B_r(x)$, which is a contradiction. Therefore $B_r(x) \subseteq F^{\mathrm{C}}$. Hence F^{C} is open, so F is closed. Since $E \subseteq F$, this implies that $\overline{E} \subseteq F$.

Now we'll show that $F \subseteq \overline{E}$. Since $E \subseteq \overline{E}$, we simply have to show that the limit points of E are contained in $\overline{E}$. So, suppose that $x \notin \overline{E}$. Since $\overline{E}$ is closed, there exists some $r > 0$ such that $B_r(x) \subseteq X\backslash\overline{E}$. Hence $B_r(x)$ contains no points of E, and therefore x cannot be a limit point of E. Hence $F \subseteq \overline{E}$.

(c) This follows by combining parts (a) and (b).

1.18 $\Rightarrow$. Suppose that M is a Banach space with respect to the norm of X. Suppose that $x_n \in M$ and $x_n \to x \in X$. Then $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in M and hence must converge to some element $y \in M$. However, as a sequence in X we then have that $x_n \to x$ and $x_n \to y$, so by uniqueness of limits, $x = y \in M$. Therefore M is closed.

$\Leftarrow$. Suppose that M is a closed subspace of X, and suppose that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in M. Then $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy in X, so there exists some $x \in X$ such that $x_n \to x$. However, M is closed, so this implies that $x \in M$. Therefore every Cauchy sequence in M converges to an element of M, so M is complete.

1.19 Suppose that $f \in C_b(\mathbf{R})$, and choose any $M > 0$. If $|f| \le M$ everywhere, then we certainly have $|f| \le M$ a.e.

For the converse, choose any $M > 0$, and suppose that there is a point where $|f(x)| > M$. Then since $|f|$ is continuous, there must be an open interval I containing x such that $|f(y)| > M$ for $y \in I$. But then $|f| > M$ on a set with positive measure, i.e., it is not true that $|f| \le M$ a.e. Hence this shows by contrapositive argument that if $|f| \le M$ a.e., then $|f| \le M$ everywhere.

Consequently,

$$\inf\{M : f(x) \le M \text{ a.e.}\} \;=\; \inf\{M : f(x) \le M \text{ for every } x\} \;=\; \sup_{x \in \mathbf{R}} |f(x)|,$$

so the uniform and L^∞ norms agree for functions in $C_b(\mathbf{R})$.

1.20 (a) Suppose that $\{x_N\}_{N \in \mathbf{N}}$ is a sequence in c and $x_N \to x$ in ℓ^∞-norm. Write $x_N = \big(x_N(k)\big)_{k \in \mathbf{N}}$ and $x = \big(x(k)\big)_{k \in \mathbf{N}}$. Since ℓ^∞ convergence implies componentwise convergence, we have that $x(k) = \lim_{N \to \infty} x_N(k)$ for each $k \in \mathbf{N}$.

By hypothesis, $y_N = \lim_{k \to \infty} x_N(k)$ exists for each N. We have

$$|y_M - y_N| \;=\; \lim_{k \to \infty} |x_M(k) - x_N(k)| \;\le\; \sup_k |x_M(k) - x_N(k)| \;=\; \|x_M - x_N\|_{\ell^\infty},$$

so $\{y_N\}_{N \in \mathbf{N}}$ is a Cauchy sequence of scalars and therefore converges, say to y.

Fix any $\varepsilon > 0$. Then there exists an N such that $\|x - x_N\|_\infty < \varepsilon$ and $|y - y_N| < \varepsilon$. Since $|x(k) - x_N(k)| < \varepsilon$ for every k, we have

$$\begin{aligned}\lim_{k \to \infty} |y - x(k)| \;&\le\; \limsup_{k \to \infty} \Big(|y - y_N| + |y_N - x_N(k)| + |x_N(k) - x(k)|\Big) \\ &\le\; \varepsilon + 0 + \varepsilon \;=\; 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, we conclude that $y = \lim_{k \to \infty} x(k)$, so $x \in c$. Thus c is closed in ℓ^∞.

Now assume in addition that $x_N \in c_0$ for each N. Then $y_N = 0$ for every N, so by the argument above we see that $y = 0$. Hence $x \in c_0$, so c_0 is closed in ℓ^∞ as well.

(b) Choose any $x = (x(1), x(2), \dots) \in c_0$. Define

$$x_N \;=\; (x(1), \dots, x(N), 0, 0, \dots).$$

Then $x_N \in c_{00}$, and

$$\lim_{N \to \infty} \|x - x_N\|_{\ell^\infty} \;=\; \lim_{N \to \infty} \sup_{k > N} |x(k)| \;=\; \limsup_{k \to \infty} |x(k)| \;=\; 0.$$

Hence c_{00} is dense in c_0. However, c_{00} is not closed, since any $x \in c_0$ with infinitely many nonzero components is an accumulation point of c_{00} but does not belong to c_{00}.

(c) Choose any $x \in c_0$. Write $x = (x(1), x(2), \dots)$, and set

$$x_N \;=\; (x(1), \dots, x(N), 0, 0, \dots) \;=\; \sum_{k=1}^{N} x(k)\,\delta_k.$$

By part (b) we know that $\|x - x_N\|_{\ell^\infty} \to 0$ as $N \to \infty$. Since the x_N is the partial sums of the series $\sum x(k)\,\delta_k$, we conclude that $x = \sum x(k)\,\delta_k$.

On the other hand, if a series $x = \sum c_k \delta_k$ converges in ℓ^∞ norm then the partial sums must converge componentwise. The partial sums are $x_N = (c_1, \dots, c_N, 0, 0, \dots)$, so the kth component of x is precisely c_k.

1.21 (a) The fact that $C_b\mathbf{R})$ is a vector space and $\|\cdot\|_\infty$ is a norm on $C_b(\mathbf{R})$ is clear, so we only need to show completeness.

Suppose that $\{f_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C_b(\mathbf{R})$ with respect to the uniform norm. Then for each x, we have

$$|f_m(x) - f_n(x)| \;\le\; \|f_m - f_n\|_\infty,$$

so $\{f_n(x)\}_{n\in\mathbf{N}}$ is a Cauchy sequence of scalars, and hence converges. Define $f(x) = \lim_{n\to\infty} f_n(x)$.

Now choose $\varepsilon > 0$. Then there exists an N such that $\|f_m - f_n\|_\infty < \varepsilon$ for all m, $n > N$. Fix $n > N$. Then for every x we have

$$|f(x) - f_n(x)| \;=\; \lim_{m\to\infty} |f_m(x) - f_n(x)| \;\le\; \|f_m - f_n\|_\infty \;\le\; \varepsilon,$$

so $\|f - f_n\|_\infty \le \varepsilon$ for all $n > N$. Also, $\|f\|_\infty \le \|f - f_n\|_\infty + \|f_n\|_\infty$, so f is bounded. Finally, the uniform limit of continuous functions is continuous, so $f \in C_b(\mathbf{R})$ and $f_n \to f$ uniformly. This shows that $C_b(\mathbf{R})$ is complete.

(b) Suppose that $f_n \in C_0(\mathbf{R})$ and $f_n \to f$ uniformly. By part (a) we have $f \in C_b(\mathbf{R})$. Given $\varepsilon > 0$, there exists some n such that $\|f - f_n\|_\infty < \varepsilon$. For this n, there exists an $R > 0$ such that $|f_n(x)| < \varepsilon$ for all $|x| > R$. Hence for $|x| > R$ we have

$$|f(x)| \;\le\; |f(x) - f_n(x)| + |f_n(x)| \;\le\; \|f - f_n\|_\infty + \varepsilon \;\le\; 2\varepsilon.$$

Hence $f(x) \to 0$ as $|x| \to \infty$, so $f \in C_0(\mathbf{R})$. Thus $C_0(\mathbf{R})$ is a closed subspace of $C_b(\mathbf{R})$.

(c) Choose any $g \in C_0(\mathbf{R})$. Then there exists an $N > 0$ such that $|g(x)| < \varepsilon$ for all $|x| > N$. Set

$$g_N(x) \;=\; \begin{cases} g(x), & |x| \le N,\\ \text{linear}, & N \le |x| \le N+1,\\ 0, & |x| > N+1. \end{cases}$$

Each g_N belongs to $C_c(\mathbf{R})$, and

$$\|g - g_N\|_\infty \;=\; \sup_{|x|>N} |g(x) - g_N(x)| \;\le\; \sup_{|x|>N} \bigl(|g(x)| + |g_N(x)|\bigr) \;\le\; 2\varepsilon,$$

so $g_N \to g$ uniformly. Hence $C_c(\mathbf{R})$ is dense in $C_0(\mathbf{R})$. However, if $g(x) = e^{-x^2}$, then g belongs to $C_0(\mathbf{R})$ but does not belong to $C_c(\mathbf{R})$, so $C_c(\mathbf{R})$ is not closed.

(d) Suppose that $f_n \in C(\mathbf{T})$ and $f_n \to f$ uniformly. By part (a) we have $f \in C_b(\mathbf{R})$. Since uniform convergence implies pointwise convergence, for each $x \in \mathbf{R}$ we have

$$f(x+1) \;=\; \lim_{n\to\infty} f_n(x+1) \;=\; \lim_{n\to\infty} f_n(x) \;=\; f(x).$$

Hence f is 1-periodic, so $f \in C(\mathbf{T})$ and therefore $C(\mathbf{T})$ is closed in $C_b(\mathbf{R})$.

1.22 (a) Let us show that $C_b^1(\mathbf{R})$ is complete. Suppose that $\{f_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C_b^1(\mathbf{R})$. Then $\{f_n\}_{n\in\mathbf{N}}$ is Cauchy in $C_b(\mathbf{R})$, so there exists an $f \in C_b(\mathbf{R})$ such that $f_n \to f$ uniformly. Additionally, by definition of $C_b^1(\mathbf{R})$, we know that

$$\|f_m' - f_n'\|_\infty \;\le\; \|f_m - f_n\|_\infty + \|f_m' - f_n'\|_\infty \;=\; \|f_m - f_n\|_{C_b^1},$$

so $\{f_n'\}_{n\in\mathbf{N}}$ is Cauchy with respect to the uniform norm. That is, $\{f_n'\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C_b(\mathbf{R})$. Since $C_b(\mathbf{R})$ is complete, there exists a $g \in C_b(\mathbf{R})$ such that $f_n' \to g$ uniformly. So, the remaining point is to show that $g = f'$, for then we will have that $f_n \to f$ in the norm of $C_b^1(\mathbf{R})$.

To see this, fix $\varepsilon > 0$. Then there exists an $N > 0$ such that $\|f_m' - f_n'\|_\infty < \varepsilon$ whenever $m, n > N$. Fix $x, y \in \mathbf{R}$ and $m, n > N$. Applying the Mean-Value Theorem to the function $f_m - f_n$, there exists a point c (depending on m, n, x, and y) between x and y such that

$$(f_m - f_n)(y) - (f_m(x) - f_n)(x) \;=\; (y-x)\,(f_m' - f_n')(c).$$

Consequently,

$$\left|\frac{f_m(y) - f_m(x)}{y-x} - \frac{f_n(y) - f_n(x)}{y-x}\right| \;=\; |f_m'(c) - f_n'(c)| \;\le\; \|f_m' - f_n'\|_\infty \;<\; \varepsilon.$$

Letting $m \to \infty$, we conclude that

$$\left|\frac{f(y) - f(x)}{y-x} - \frac{f_n(y) - f_n(x)}{y-x}\right| \;\le\; \varepsilon.$$

This is valid for $x, y \in \mathbf{R}$ as long as $n > N$.

Now, since f_n is differentiable, there exists a $\delta > 0$ such that

$$|x-y| < \delta \;\implies\; \left|f_n'(x) - \frac{f_n(y) - f_n(x)}{y-x}\right| \;<\; \varepsilon.$$

Further, since $f_n' \to g$ uniformly, there exists an M such that $\|f_n' - g\|_\infty < \varepsilon$ whenever $n > M$. Fix x, and suppose that $|x-y| < \delta$. Then for $n > M, N$ we have

$$\begin{aligned} \left| g(x) - \frac{f(y)-f(x)}{y-x} \right| &< |g(x) - f_n'(x)| + \left| f_n'(x) - \frac{f_n(y)-f_n(x)}{y-x} \right| \\ &\quad + \left| \frac{f_n(y)-f_n(x)}{y-x} - \frac{f(y)-f(x)}{y-x} \right| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Hence

$$g(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x},$$

so f is differentiable at x, and $f'(x) = g(x)$. Thus $f_n \to f$ in the norm of $C_b^1(\mathbf{R})$, so this space is complete.

A proof by induction shows that $C_b^m(\mathbf{R})$ is complete for each m.

(b) If we replace the norm on $C_b^1(\mathbf{R})$ by the uniform norm, then it is no longer complete. Let $w(x) = \max\{1-|x|, 0\}$ be the hat function on $[-1,1]$. Then we can find differentiable functions $f_n \in C_b^1(\mathbf{R})$ such that $\|w - f_n\|_\infty \to 0$. For example, we just need to "smooth out" the corners of the graph of w to find f_n. Therefore $\{f_n\}$ is a Cauchy sequence in the uniform norm, but it does not converge *within* $C_b^1(\mathbf{R})$ because $w \notin C_b^1(\mathbf{R})$.

1.23 (a) If f is Hölder continuous with $\alpha > 0$ then

$$\lim_{y\to x} \left| \frac{f(x)-f(y)}{x-y} \right| \le \lim_{y\to x} \frac{C\,|x-y|^\alpha}{|x-y|} = \lim_{y\to x} C\,|x-y|^{1-\alpha} = 0.$$

Therefore f is differentiable and $f'(x) = 0$ for every x, so f is constant.

(b) By the Mean-Value Theorem, given x and y there exists some c between x and y such that $f(x) - f(y) = f'(c)\,(x-y)$, so

$$|f(x) - f(y)| = |f'(c)|\,|x-y| \le \|f'\|_\infty\,|x-y|.$$

If f' is bounded, then it follows that f is Lipschitz.

The function $f(x) = |x|$ is Lipschitz, but is not differentiable at every point.

(c) By definition, $0 \le \|f\|_{C^\alpha} < \infty$ for each $f \in C^\alpha(\mathbf{R})$.

Suppose that $\|f\|_{C^\alpha} = 0$. Then $f(0) = 0$ and

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0, \qquad \text{all } x \ne y.$$

Consequently, $f(x) = f(y)$ for all $x \ne y$. Hence $f(x) = 0$ for every x, i.e., $f = 0$.