Elementary LINEAR ALGEBRA

8e

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Ron Larson

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Elementary Linear Algebra

Elementary Linear Algebra

8e

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Eighth Edition

Ron Larson

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Contents

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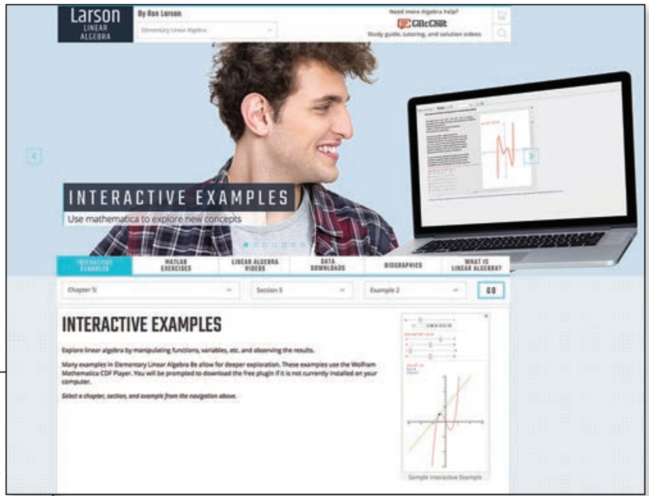
Preface

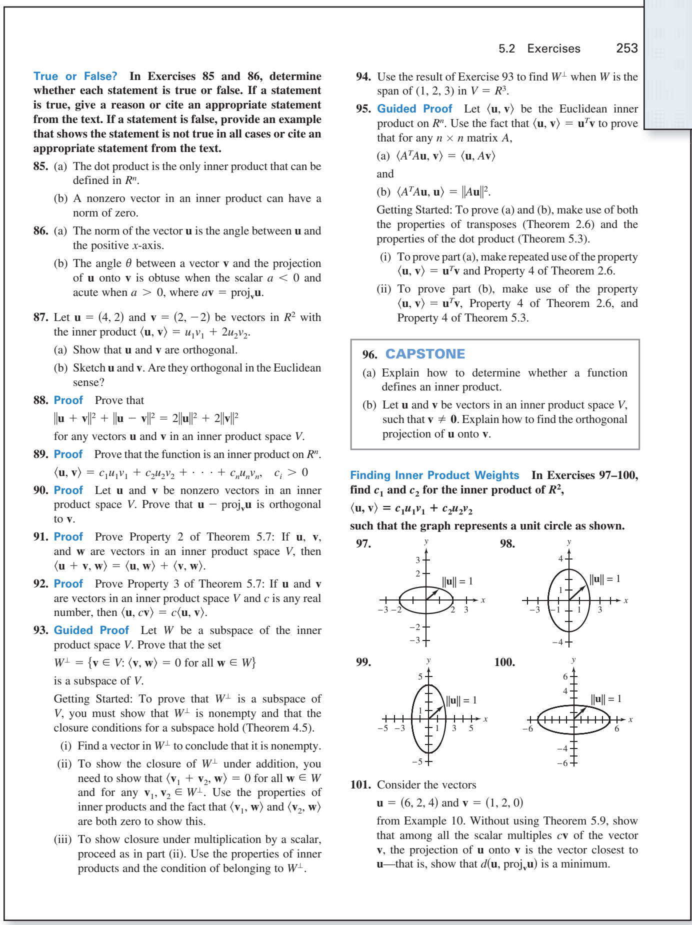
Welcome to *Elementary Linear Algebra*, Eighth Edition. I am proud to present to you this new edition. As with all editions, I have been able to incorporate many useful comments from you, our user. And while much has changed in this revision, you will still find what you expect—a pedagogically sound, mathematically precise, and comprehensive textbook. Additionally, I am pleased and excited to offer you something brand new— a companion website at **LarsonLinearAlgebra.com.** My goal for every edition of this textbook is to provide students with the tools that they need to master linear algebra. I hope you find that the changes in this edition, together with **LarsonLinearAlgebra.com,** will help accomplish just that.

New To This Edition

NEW *LarsonLinearAlgebra.com*

This companion website offers multiple tools and resources to supplement your learning. Access to these features is *free*. Watch videos explaining concepts from the book, explore examples, download data sets and much more.





5.2 Exercises 253

True or False? **In Exercises 85 and 86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

85. (a) The dot product is the only inner product that can be defined in R^n.

(b) A nonzero vector in an inner product can have a norm of zero.

86. (a) The norm of the vector $\mathbf{u}$ is the angle between $\mathbf{u}$ and the positive x-axis.

(b) The angle θ between a vector $\mathbf{v}$ and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is obtuse when the scalar $a < 0$ and acute when $a > 0$, where $a\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{u}$.

87. Let $\mathbf{u} = (4, 2)$ and $\mathbf{v} = (2, -2)$ be vectors in R^2 with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$.

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) Sketch $\mathbf{u}$ and $\mathbf{v}$. Are they orthogonal in the Euclidean sense?

88. **Proof** Prove that

$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$

for any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V.

89. **Proof** Prove that the function is an inner product on R^n.

$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n, \quad c_i > 0$

90. **Proof** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in an inner product space V. Prove that $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.

91. **Proof** Prove Property 2 of Theorem 5.7: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in an inner product space V, then $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

92. **Proof** Prove Property 3 of Theorem 5.7: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V and c is any real number, then $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

93. **Guided Proof** Let W be a subspace of the inner product space V. Prove that the set

$W^{\perp} = \{\mathbf{v} \in V: \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}$

is a subspace of V.

Getting Started: To prove that $W^{\perp}$ is a subspace of V, you must show that $W^{\perp}$ is nonempty and that the closure conditions for a subspace hold (Theorem 4.5).

(i) Find a vector in $W^{\perp}$ to conclude that it is nonempty.

(ii) To show the closure of $W^{\perp}$ under addition, you need to show that $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$ and for any $\mathbf{v}_1, \mathbf{v}_2 \in W^{\perp}$. Use the properties of inner products and the fact that $\langle \mathbf{v}_1, \mathbf{w} \rangle$ and $\langle \mathbf{v}_2, \mathbf{w} \rangle$ are both zero to show this.

(iii) To show closure under multiplication by a scalar, proceed as in part (ii). Use the properties of inner products and the condition of belonging to $W^{\perp}$.

94. Use the result of Exercise 93 to find $W^{\perp}$ when W is the span of (1, 2, 3) in $V = R^3$.

95. **Guided Proof** Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on R^n. Use the fact that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$ to prove that for any $n \times n$ matrix A,

(a) $\langle A^TA\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$

and

(b) $\langle A^TA\mathbf{u}, \mathbf{u} \rangle = \|A\mathbf{u}\|^2$.

Getting Started: To prove (a) and (b), make use of both the properties of transposes (Theorem 2.6) and the properties of the dot product (Theorem 5.3).

(i) To prove part (a), make repeated use of the property $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$ and Property 4 of Theorem 2.6.

(ii) To prove part (b), make use of the property $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$, Property 4 of Theorem 2.6, and Property 4 of Theorem 5.3.

96. **CAPSTONE**

(a) Explain how to determine whether a function defines an inner product.

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Explain how to find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

Finding Inner Product Weights **In Exercises 97–100, find c_1 and c_2 for the inner product of R^2,**

$\langle \mathbf{u}, \mathbf{v} \rangle = c_1u_1v_1 + c_2u_2v_2$

such that the graph represents a unit circle as shown.

97. **98.** **99.** **100.**

101. Consider the vectors

$\mathbf{u} = (6, 2, 4)$ and $\mathbf{v} = (1, 2, 0)$

from Example 10. Without using Theorem 5.9, show that among all the scalar multiples $c\mathbf{v}$ of the vector $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector closest to $\mathbf{u}$—that is, show that $d(\mathbf{u}, \text{proj}_{\mathbf{v}}\mathbf{u})$ is a minimum.

REVISED *Exercise Sets*

The exercise sets have been carefully and extensively examined to ensure they are rigorous, relevant, and cover all the topics necessary to understand the fundamentals of linear algebra. The exercises are ordered and titled so you can see the connections between examples and exercises. Many new skill-building, challenging, and application exercises have been added. As in earlier editions, the following pedagogically-proven types of exercises are included.

- **True or False Exercises**
- **Proofs**
- **Guided Proofs**
- **Writing Exercises**
- **Technology Exercises** (indicated throughout the text with)

Exercises utilizing **electronic data sets** are indicated by and found at **CengageBrain.com.**

Table of Contents Changes

Based on market research and feedback from users, Section 2.5 in the previous edition (Applications of Matrix Operations) has been expanded from one section to two sections to include content on Markov chains. So now, Chapter 2 has *two* application sections: Section 2.5 (Markov Chains) and Section 2.6 (More Applications of Matrix Operations). In addition, Section 7.4 (Applications of Eigenvalues and Eigenvectors) has been expanded to include content on constrained optimization.

Trusted Features

CalcChat®

For the past several years, an independent website—CalcChat.com—has provided free solutions to all odd-numbered problems in the text. Thousands of students have visited the site for practice and help with their homework from live tutors. You can also use your smartphone's QR Code® reader to scan the icon at the beginning of each exercise set to access the solutions.


2 Matrices

2.1 Operations with Matrices
2.2 Properties of Matrix Operations
2.3 The Inverse of a Matrix
2.4 Elementary Matrices
2.5 Markov Chains
2.6 More Applications of Matrix Operations

Data Encryption (p. 94)

Computational Fluid Dynamics (p. 79)

Beam Deflection (p. 64)

Information Retrieval (p. 58)

Flight Crew Scheduling (p. 47)

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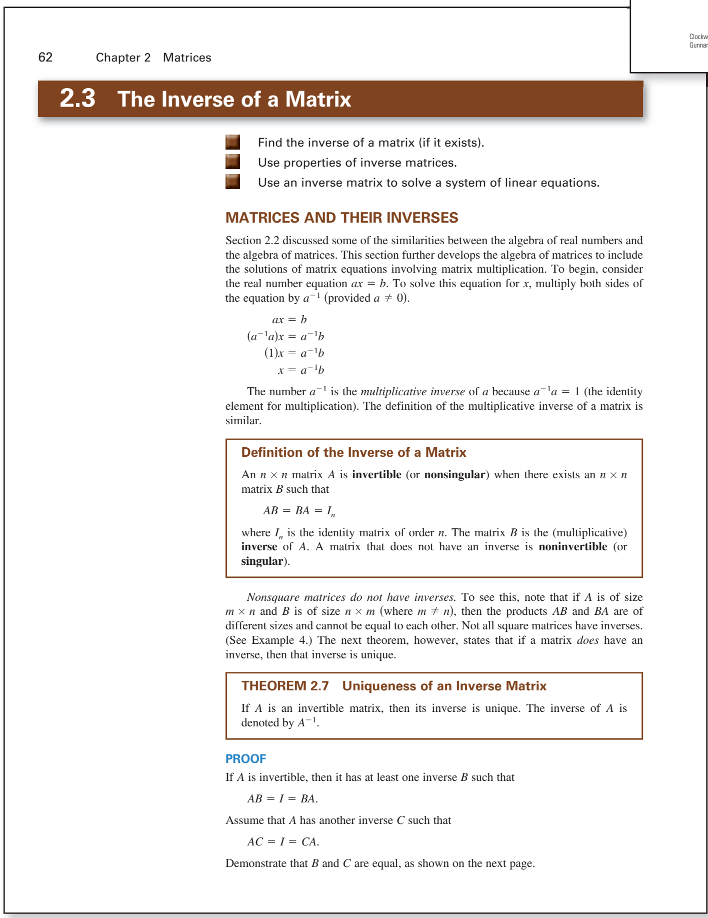
62 Chapter 2 Matrices

2.3 The Inverse of a Matrix

- Find the inverse of a matrix (if it exists).
- Use properties of inverse matrices.
- Use an inverse matrix to solve a system of linear equations.

MATRICES AND THEIR INVERSES

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply both sides of the equation by a^{-1} (provided $a \neq 0$).

$$ax = b$$
$$(a^{-1}a)x = a^{-1}b$$
$$(1)x = a^{-1}b$$
$$x = a^{-1}b$$

The number a^{-1} is the *multiplicative inverse* of a because $a^{-1}a = 1$ (the identity element for multiplication). The definition of the multiplicative inverse of a matrix is similar.

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is the (multiplicative) **inverse** of A. A matrix that does not have an inverse is **noninvertible** (or **singular**).

Nonsquare matrices do not have inverses. To see this, note that if A is of size $m \times n$ and B is of size $n \times m$ (where $m \neq n$), then the products AB and BA are of different sizes and cannot be equal to each other. Not all square matrices have inverses. (See Example 4.) The next theorem, however, states that if a matrix *does* have an inverse, then that inverse is unique.

THEOREM 2.7 Uniqueness of an Inverse Matrix

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

PROOF

If A is invertible, then it has at least one inverse B such that

$$AB = I = BA.$$

Assume that A has another inverse C such that

$$AC = I = CA.$$

Demonstrate that B and C are equal, as shown on the next page.

Chapter Openers

Each *Chapter Opener* highlights five real-life applications of linear algebra found throughout the chapter. Many of the applications reference the *Linear Algebra Applied* feature (discussed on the next page). You can find a full list of the applications in the *Index of Applications* on the inside front cover.

Section Objectives

A bulleted list of learning objectives, located at the beginning of each section, provides you the opportunity to preview what will be presented in the upcoming section.

Theorems, Definitions, and Properties

Presented in clear and mathematically precise language, all theorems, definitions, and properties are highlighted for emphasis and easy reference.

Proofs in Outline Form

In addition to proofs in the exercises, some proofs are presented in outline form. This omits the need for burdensome calculations.

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Discovery

Using the *Discovery* feature helps you develop an intuitive understanding of mathematical concepts and relationships.

Technology Notes

Technology notes show how you can use graphing utilities and software programs appropriately in the problem-solving process. Many of the *Technology* notes reference the **Technology Guide** at **CengageBrain.com.**

EXAMPLE 4 **Finding a Transition Matrix**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the transition matrix from B to B' for the bases for R^3 below.

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \quad \text{and} \quad B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

SOLUTION

First use the vectors in the two bases to form the matrices B and B'.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Then form the matrix $[B' \quad B]$ and use Gauss-Jordan elimination to rewrite $[B' \quad B]$ as $[I_3 \quad P^{-1}]$.

$$\left[\begin{array}{rrrrrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrrrr} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$$

From this, you can conclude that the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}.$$

Multiply P^{-1} by the coordinate matrix of $\mathbf{x} = [1 \quad 2 \quad -1]^T$ to see that the result is the same as that obtained in Example 3.

DISCOVERY

1. Let $B = \{(1, 0), (1, 2)\}$ and $B' = \{(1, 0), (0, 1)\}$. Form the matrix $[B' \quad B]$.

2. Make a conjecture about the necessity of using Gauss-Jordan elimination to obtain the transition matrix P^{-1} when the change of basis is from a nonstandard basis to a standard basis.

SOLUTION

Notice that three of the entries in th[e] … the work in the expansion, use the …

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{3…}$$

The cofactors C_{23}, C_{33}, and C_{43} have zero coefficients, so you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \qquad \text{Delete 1st row and 3rd column.}$$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \qquad \text{Simplify.}$$

Expanding by cofactors in the second row yields

$$C_{13} = (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + \ldots$$

$$= 0 + 2(1)(-4) + 3(-1)(-7)$$

$$= 13.$$

You obtain

$$|A| = 3(13)$$

$$= 39.$$

TECHNOLOGY

Many graphing utilities and software programs can find the determinant of a square matrix. If you use a graphing utility, then you may see something similar to the screen below for Example 4. The **Technology Guide** at *CengageBrain.com* can help you use technology to find a determinant.

```
A
    [[1  -2  3  0 ]
     [-1  1  0  2 ]
     [0   2  0  3 ]
     [3   4  0  -2]]
det A
                 39
```

LINEAR ALGEBRA APPLIED

Time-frequency analysis of irregular physiological signals, such as beat-to-beat cardiac rhythm variations (also known as heart rate variability or HRV), can be difficult. This is because the structure of a signal can include multiple periodic, nonperiodic, and pseudo-periodic components. Researchers have proposed and validated a simplified HRV analysis method called orthonormal-basis partitioning and time-frequency representation (OPTR). This method can detect both abrupt and slow changes in the HRV signal's structure, divide a nonstationary HRV signal into segments that are "less nonstationary," and determine patterns in the HRV. The researchers found that although it had poor time resolution with signals that changed gradually, the OPTR method accurately represented multicomponent and abrupt changes in both real-life and simulated HRV signals. *(Source: Orthonormal-Basis Partitioning and Time-Frequency Representation of Cardiac Rhythm Dynamics, Aysin, Benhur, et al, IEEE Transactions on Biomedical Engineering, 52, no. 5)*

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2 Projects

	Test 1	Test 2
Anna	84	96
Bruce	56	72
Chris	78	83
David	82	91

1 Exploring Matrix Multiplication

The table shows the first two test scores for Anna, Bruce, Chris, and David. Use the table to create a matrix M to represent the data. Input M into a software program or a graphing utility and use it to answer the questions below.

1. Which test was more difficult? Which was easier? Explain.
2. How would you rank the performances of the four students?
3. Describe the meanings of the matrix products $M\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $M\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
4. Describe the meanings of the matrix products $[1 \quad 0 \quad 0 \quad 0]M$ and $[0 \quad 0 \quad 1 \quad 0]M$.
5. Describe the meanings of the matrix products $M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{2}M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
6. Describe the meanings of the matrix products $[1 \quad 1 \quad 1 \quad 1]M$ and $\frac{1}{4}[1 \quad 1 \quad 1 \quad 1]M$.
7. Describe the meaning of the matrix product $[1 \quad 1 \quad 1 \quad 1]M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
8. Use matrix multiplication to find the combined overall average score on both tests.
9. How could you use matrix multiplication to scale the scores on test 1 by a factor of 1.1?

2 Nilpotent Matrices

Let A be a nonzero square matrix. Is it possible that a positive integer k exists such that $A^k = O$? For example, find A^3 for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A square matrix A is **nilpotent of index k** when $A \neq O$, $A^2 \neq O$, . . . , $A^{k-1} \neq O$, but $A^k = O$. In this project you will explore nilpotent matrices.

1. The matrix in the example above is nilpotent. What is its index?
2. Use a software program or a graphing utility to determine which matrices below are nilpotent and find their indices.

   (a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$
3. Find 3×3 nilpotent matrices of indices 2 and 3.
4. Find 4×4 nilpotent matrices of indices 2, 3, and 4.
5. Find a nilpotent matrix of index 5.
6. Are nilpotent matrices invertible? Prove your answer.
7. When A is nilpotent, what can you say about A^T? Prove your answer.
8. Show that if A is nilpotent, then $I - A$ is invertible.

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Linear Algebra Applied

The *Linear Algebra Applied* feature describes a real-life application of concepts discussed in a section. These applications include biology and life sciences, business and economics, engineering and technology, physical sciences, and statistics and probability.

Capstone Exercises

The *Capstone* is a conceptual problem that synthesizes key topics to check students' understanding of the section concepts. I recommend it.

Chapter Projects

Two per chapter, these offer the opportunity for group activities or more extensive homework assignments, and are focused on theoretical concepts or applications. Many encourage the use of technology.

Instructor Resources

Media

Instructor's Solutions Manual
The *Instructor's Solutions Manual* provides worked-out solutions for all even-numbered exercises in the text.

Cengage Learning Testing Powered by Cognero (ISBN: 978-1-305-65806-6) is a flexible, online system that allows you to author, edit, and manage test bank content, create multiple test versions in an instant, and deliver tests from your LMS, your classroom, or wherever you want. This is available online at **cengage.com/login.**

Turn the Light On with MindTap for Larson's *Elementary Linear Algebra*
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Student Resources

Print

Student Solutions Manual
ISBN-13: 978-1-305-87658-3
The *Student Solutions Manual* provides complete worked-out solutions to all odd-numbered exercises in the text. Also included are the solutions to all Cumulative Test problems.

Media

MindTap for Larson's *Elementary Linear Algebra*
MindTap is a digital representation of your course that provides you with the tools you need to better manage your limited time, stay organized and be successful. You can complete assignments whenever and wherever you are ready to learn with course material specially customized for you by your instructor and streamlined in one proven, easy-to-use interface. With an array of study tools, you'll get a true understanding of course concepts, achieve better grades and set the groundwork for your future courses.

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To access additional course materials and companion resources, please visit **CengageBrain.com.** At the **CengageBrain.com** home page, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where free companion resources can be found.

Acknowledgements

I would like to thank the many people who have helped me during various stages of writing this new edition. In particular, I appreciate the feedback from the dozens of instructors who took part in a detailed survey about how they teach linear algebra. I also appreciate the efforts of the following colleagues who have provided valuable suggestions throughout the life of this text:

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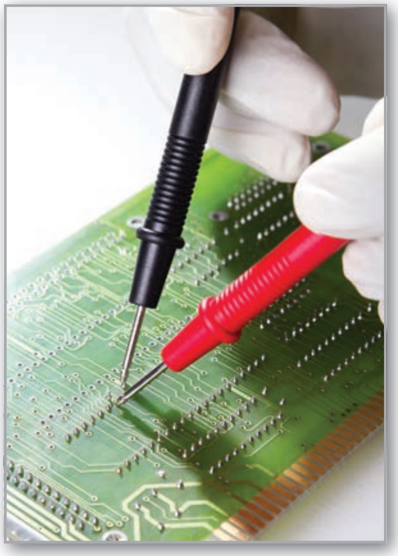
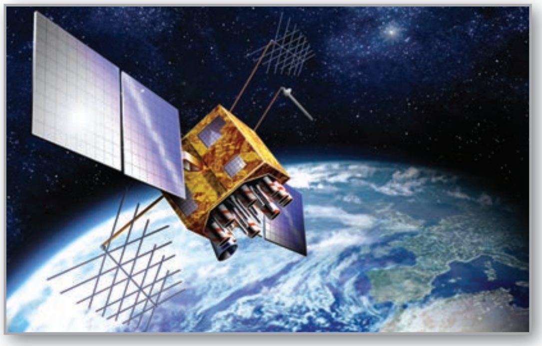
1 Systems of Linear Equations



Electrical Network Analysis (p. 30)

Traffic Flow (p. 28)



Global Positioning System (p. 16)

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1.1 Introduction to Systems of Linear Equations

- Recognize a linear equation in n variables.
- Find a parametric representation of a solution set.
- Determine whether a system of linear equations is consistent or inconsistent.
- Use back-substitution and Gaussian elimination to solve a system of linear equations.

LINEAR EQUATIONS IN *n* VARIABLES

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally, you will find examples and exercises requiring a knowledge of calculus, and these are marked in the text.

Early in your study of linear algebra, you will discover that many of the solution methods involve multiple arithmetic steps, so it is essential that you check your work. Use software or a calculator to check your work and perform routine computations.

Although you will be familiar with some material in this chapter, you should carefully study the methods presented. This will cultivate and clarify your intuition for the more abstract material that follows.

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \qquad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** x and y. Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \qquad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

This is a **linear equation in three variables** x, y, and z. A linear equation in n variables is defined below.

Definition of a Linear Equation in *n* Variables

A **linear equation in *n* variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient,** and x_1 is the **leading variable.**

Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

EXAMPLE 1 Linear and Nonlinear Equations

Each equation is linear.

a. $3x + 2y = 7$ **b.** $\frac{1}{2}x + y - \pi z = \sqrt{2}$ **c.** $(\sin \pi)x_1 - 4x_2 = e^2$

Each equation is not linear.

a. $xy + z = 2$ **b.** $e^x - 2y = 4$ **c.** $\sin x_1 + 2x_2 - 3x_3 = 0$

SOLUTIONS AND SOLUTION SETS

A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, s_3, \ldots, s_n$ that satisfy the equation when you substitute the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \ldots, \quad x_n = s_n$$

into the equation. For example, $x_1 = 2$ and $x_2 = 1$ satisfy the equation $x_1 + 2x_2 = 4$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is its **solution set,** and when you have found this set, you have **solved** the equation. To describe the entire solution set of a linear equation, use a **parametric representation,** as illustrated in Examples 2 and 3.

EXAMPLE 2 Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

SOLUTION

To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. Solving for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable x_2 is **free,** which means that it can take on any real value. The variable x_1 is not free because its value depends on the value assigned to x_2. To represent the infinitely many solutions of this equation, it is convenient to introduce a third variable t called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \quad x_2 = t, \quad t \text{ is any real number.}$$

To obtain particular solutions, assign values to the parameter t. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$.

To parametrically represent the solution set of the linear equation in Example 2 another way, you could have chosen x_1 to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \quad x_2 = 2 - \tfrac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, when an equation has more than one free variable, choose the variables that occur last in the equation to be the free variables.

EXAMPLE 3 Parametric Representation of a Solution Set

Solve the linear equation $3x + 2y - z = 3$.

SOLUTION

Choosing y and z to be the free variables, solve for x to obtain

$$3x = 3 - 2y + z$$
$$x = 1 - \tfrac{2}{3}y + \tfrac{1}{3}z.$$

Letting $y = s$ and $z = t$, you obtain the parametric representation

$$x = 1 - \tfrac{2}{3}s + \tfrac{1}{3}t, \quad y = s, \quad z = t$$

where s and t are any real numbers. Two particular solutions are

$$x = 1, y = 0, z = 0 \quad \text{and} \quad x = 1, y = 1, z = 2.$$

SYSTEMS OF LINEAR EQUATIONS

A **system of *m* linear equations in *n* variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

REMARK

The double-subscript notation indicates a_{ij} is the coefficient of x_j in the ith equation.

A system of linear equations is also called a **linear system.** A **solution** of a linear system is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each equation in the system. For example, the system

$$\begin{aligned}
3x_1 + 2x_2 &= 3\\
-x_1 + x_2 &= 4
\end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy *both* equations. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

DISCOVERY

1. Graph the two lines

$$\begin{aligned}
3x - y &= 1\\
2x - y &= 0
\end{aligned}$$

in the xy-plane. Where do they intersect? How many solutions does this system of linear equations have?

2. Repeat this analysis for the pairs of lines

$$\begin{aligned}
3x - y &= 1\\
3x - y &= 0
\end{aligned} \quad \text{and} \quad \begin{aligned}
3x - y &= 1\\
6x - 2y &= 2.
\end{aligned}$$

3. What basic types of solution sets are possible for a system of two linear equations in two variables?

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.

LINEAR ALGEBRA APPLIED

In a chemical reaction, atoms reorganize in one or more substances. For example, when methane gas (CH_4) combines with oxygen (O_2) and burns, carbon dioxide (CO_2) and water (H_2O) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O.$$

A chemical reaction can neither create nor destroy atoms. So, all of the atoms represented on the left side of the arrow must also be on the right side of the arrow. This is called *balancing* the chemical equation. In the above example, chemists can use a system of linear equations to find values of x_1, x_2, x_3, and x_4 that will balance the chemical equation.

Elnur/Shutterstock.com

It is possible for a system of linear equations to have exactly one solution, infinitely many solutions, or no solution. A system of linear equations is **consistent** when it has at least one solution and **inconsistent** when it has no solution.

EXAMPLE 4 Systems of Two Equations in Two Variables

Solve and graph each system of linear equations.

a. $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$ **b.** $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ **c.** $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

SOLUTION

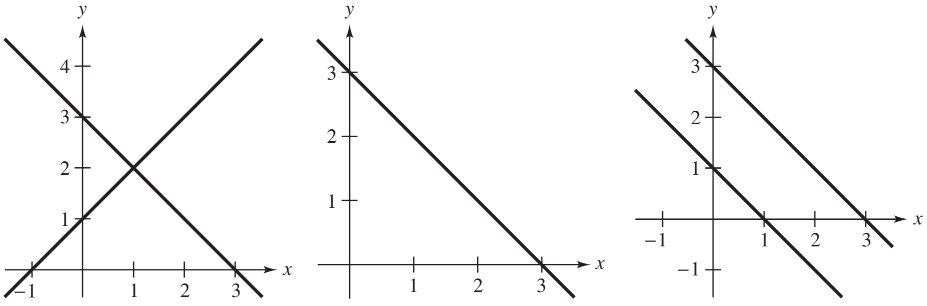
a. This system has exactly one solution, $x = 1$ and $y = 2$. One way to obtain the solution is to add the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is two *intersecting* lines, as shown in Figure 1.1(a).

b. This system has infinitely many solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is

$$x = 3 - t, \quad y = t, \quad t \text{ is any real number.}$$

The graph of this system is two *coincident* lines, as shown in Figure 1.1(b).

c. This system has no solution because the sum of two numbers cannot be 3 and 1 simultaneously. The graph of this system is two *parallel* lines, as shown in Figure 1.1(c).



a. Two intersecting lines: $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$

b. Two coincident lines: $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$

c. Two parallel lines: $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

Figure 1.1

Example 4 illustrates the three basic types of solution sets that are possible for a system of linear equations. This result is stated here without proof. (The proof is provided later in Theorem 2.5.)

Number of Solutions of a System of Linear Equations

For a system of linear equations, precisely one of the statements below is true.

1. The system has exactly one solution (consistent system).
2. The system has infinitely many solutions (consistent system).
3. The system has no solution (inconsistent system).

SOLVING A SYSTEM OF LINEAR EQUATIONS

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad\qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. To solve such a system, use **back-substitution.**

EXAMPLE 5 Using Back-Substitution in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{aligned} x - 2y &= 5 && \text{Equation 1} \\ y &= -2 && \text{Equation 2} \end{aligned}$$

SOLUTION

From Equation 2, you know that $y = -2$. By substituting this value of y into Equation 1, you obtain

$$\begin{aligned} x - 2(-2) &= 5 && \text{Substitute } -2 \text{ for } y. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The system has exactly one solution: $x = 1$ and $y = -2$.

The term *back-substitution* implies that you work *backwards*. For instance, in Example 5, the second equation gives you the value of y. Then you substitute that value into the first equation to solve for x. Example 6 further demonstrates this procedure.

EXAMPLE 6 Using Back-Substitution in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 && \text{Equation 1} \\ y + 3z &= 5 && \text{Equation 2} \\ z &= 2 && \text{Equation 3} \end{aligned}$$

SOLUTION

From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

$$\begin{aligned} y + 3(2) &= 5 && \text{Substitute 2 for } z. \\ y &= -1. && \text{Solve for } y. \end{aligned}$$

Then, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \text{Substitute } -1 \text{ for } y \text{ and 2 for } z. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

Two systems of linear equations are **equivalent** when they have the same solution set. To solve a system that is not in row-echelon form, first rewrite it as an *equivalent* system that is in row-echelon form using the operations listed on the next page.

Carl Friedrich Gauss (1777–1855)

German mathematician Carl Friedrich Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. Gauss used a form of what is now known as Gaussian elimination in his research. Although this method was named in his honor, the Chinese used an almost identical method some 2000 years prior to Gauss.

Operations That Produce Equivalent Systems

Each of these operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, using one of the three basic operations to obtain each system. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 7 Using Elimination to Rewrite a System in Row-Echelon Form

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION

Although there are several ways to begin, you want to use a systematic procedure that can be applied to larger systems. Work from the upper left corner of the system, saving the x at the upper left and eliminating the other x-terms from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that you have eliminated all but the first x from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \quad y = -1, \quad z = 2.$$

Each of the three equations in Example 7 represents a plane in a three-dimensional coordinate system. The unique solution of the system is the point $(x, y, z) = (1, -1, 2)$, so the three planes intersect at this point, as shown in Figure 1.2.

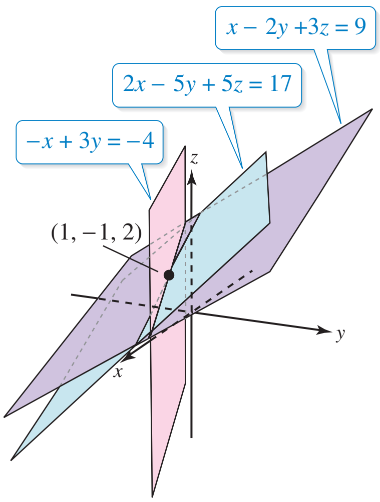


Figure 1.2

Many steps are often required to solve a system of linear equations, so it is very easy to make arithmetic errors. You should develop the habit of *checking your solution by substituting it into each equation in the original system.* For instance, in Example 7, check the solution $x = 1$, $y = -1$, and $z = 2$ as shown below.

$$\begin{aligned}
&\text{Equation 1:} && (1) - 2(-1) + 3(2) = 9\\
&\text{Equation 2:} && -(1) + 3(-1) = -4\\
&\text{Equation 3:} && 2(1) - 5(-1) + 5(2) = 17
\end{aligned}$$

Substitute the solution into each equation of the original system.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage of the Gaussian elimination process, you obtain a false statement such as $0 = -2$.

EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
2x_1 - x_2 - 2x_3 &= 2\\
x_1 + 2x_2 - 3x_3 &= -1
\end{aligned}$$

SOLUTION

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
x_1 + 2x_2 - 3x_3 &= -1
\end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
5x_2 - 4x_3 &= -2
\end{aligned}$$

Adding -1 times the first equation to the third equation produces a new third equation.

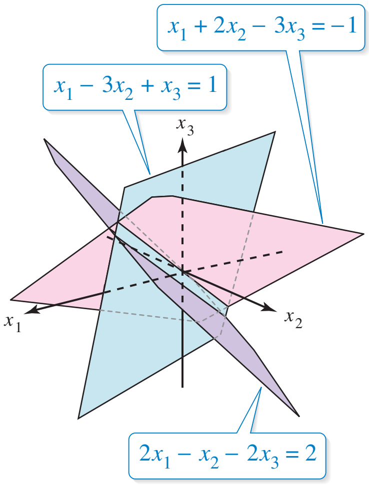
(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.)

$$\begin{aligned}
x_1 - 3x_2 + x_3 &= 1\\
5x_2 - 4x_3 &= 0\\
0 &= -2
\end{aligned}$$

Subtracting the second equation from the third equation produces a new third equation.

The statement $0 = -2$ is false, so this system has no solution. Moreover, this system is equivalent to the original system, so the original system also has no solution.

As in Example 7, the three equations in Example 8 represent planes in a three-dimensional coordinate system. In this example, however, the system is inconsistent. So, the planes do not have a point in common, as shown at the right.



This section ends with an example of a system of linear equations that has infinitely many solutions. You can represent the solution set for such a system in parametric form, as you did in Examples 2 and 3.

EXAMPLE 9 **A System with Infinitely Many Solutions**

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 \quad\quad - 3x_3 &= -1 \\ -x_1 + 3x_2 \quad\quad &= 1 \end{aligned}$$

SOLUTION

Begin by rewriting the system in row-echelon form, as shown below.

$$\begin{aligned} x_1 \quad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 \quad\quad &= 1 \end{aligned}$$

Interchange the first two equations.

$$\begin{aligned} x_1 \quad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 \quad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Adding -3 times the second equation to the third equation eliminates the third equation.

The third equation is unnecessary, so omit it to obtain the system shown below.

$$\begin{aligned} x_1 \quad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$x_1 = 3t - 1, \quad x_2 = t, \quad x_3 = t, \quad t$ is any real number.

DISCOVERY

1. Graph the two lines represented by the system of equations.

$$\begin{aligned} x - 2y &= 1 \\ -2x + 3y &= -3 \end{aligned}$$

2. Use Gaussian elimination to solve this system as shown below.

$$\begin{aligned} x - 2y &= 1 \\ -1y &= -1 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

Graph the system of equations you obtain at each step of this process. What do you observe about the lines?

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.

REMARK

You are asked to repeat this graphical analysis for other systems in Exercises 91 and 92.

1.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.

Linear Equations In Exercises 1–6, determine whether the equation is linear in the variables x and y.

1. $2x - 3y = 4$
2. $3x - 4xy = 0$
3. $\frac{3}{y} + \frac{2}{x} - 1 = 0$
4. $x^2 + y^2 = 4$
5. $2 \sin x - y = 14$
6. $(\cos 3)x + y = -16$

Parametric Representation In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

7. $2x - 4y = 0$
8. $3x - \frac{1}{2}y = 9$
9. $x + y + z = 1$
10. $12x_1 + 24x_2 - 36x_3 = 12$

Graphical Analysis In Exercises 11–24, graph the system of linear equations. Solve the system and interpret your answer.

11. $\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$
12. $\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$
13. $\begin{aligned} -x + y &= 1 \\ 3x - 3y &= 4 \end{aligned}$
14. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$
15. $\begin{aligned} 3x - 5y &= 7 \\ 2x + y &= 9 \end{aligned}$
16. $\begin{aligned} -x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$
17. $\begin{aligned} 2x - y &= 5 \\ 5x - y &= 11 \end{aligned}$
18. $\begin{aligned} x - 5y &= 21 \\ 6x + 5y &= 21 \end{aligned}$
19. $\begin{aligned} \frac{x+3}{4} + \frac{y-1}{3} &= 1 \\ 2x - y &= 12 \end{aligned}$
20. $\begin{aligned} \frac{x-1}{2} + \frac{y+2}{3} &= 4 \\ x - 2y &= 5 \end{aligned}$
21. $\begin{aligned} 0.05x - 0.03y &= 0.07 \\ 0.07x + 0.02y &= 0.16 \end{aligned}$
22. $\begin{aligned} 0.2x - 0.5y &= -27.8 \\ 0.3x - 0.4y &= 68.7 \end{aligned}$
23. $\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$
24. $\begin{aligned} \frac{2x}{3} + \frac{y}{6} &= \frac{2}{3} \\ 4x + y &= 4 \end{aligned}$

Back-Substitution In Exercises 25–30, use back-substitution to solve the system.

25. $\begin{aligned} x_1 - x_2 &= 2 \\ x_2 &= 3 \end{aligned}$
26. $\begin{aligned} 2x_1 - 4x_2 &= 6 \\ 3x_2 &= 9 \end{aligned}$
27. $\begin{aligned} -x + y - z &= 0 \\ 2y + z &= 3 \\ \tfrac{1}{2}z &= 0 \end{aligned}$
28. $\begin{aligned} x - y &= 5 \\ 3y + z &= 11 \\ 4z &= 8 \end{aligned}$
29. $\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$
30. $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$

The symbol indicates an exercise in which you are instructed to use a graphing utility or software program.

Graphical Analysis In Exercises 31–36, complete parts (a)–(e) for the system of equations.

(a) Use a graphing utility to graph the system.

(b) Use the graph to determine whether the system is consistent or inconsistent.

(c) If the system is consistent, approximate the solution.

(d) Solve the system algebraically.

(e) Compare the solution in part (d) with the approximation in part (c). What can you conclude?

31. $\begin{aligned} -3x - y &= 3 \\ 6x + 2y &= 1 \end{aligned}$
32. $\begin{aligned} 4x - 5y &= 3 \\ -8x + 10y &= 14 \end{aligned}$
33. $\begin{aligned} 2x - 8y &= 3 \\ \tfrac{1}{2}x + y &= 0 \end{aligned}$
34. $\begin{aligned} 9x - 4y &= 5 \\ \tfrac{1}{2}x + \tfrac{1}{3}y &= 0 \end{aligned}$
35. $\begin{aligned} 4x - 8y &= 9 \\ 0.8x - 1.6y &= 1.8 \end{aligned}$
36. $\begin{aligned} -14.7x + 2.1y &= 1.05 \\ 44.1x - 6.3y &= -3.15 \end{aligned}$

System of Linear Equations In Exercises 37–56, solve the system of linear equations.

37. $\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$
38. $\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$
39. $\begin{aligned} 3u + v &= 240 \\ u + 3v &= 240 \end{aligned}$
40. $\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$
41. $\begin{aligned} 9x - 3y &= -1 \\ \tfrac{1}{5}x + \tfrac{2}{5}y &= -\tfrac{1}{3} \end{aligned}$
42. $\begin{aligned} \tfrac{2}{3}x_1 + \tfrac{1}{6}x_2 &= 0 \\ 4x_1 + x_2 &= 0 \end{aligned}$
43. $\begin{aligned} \frac{x-2}{4} + \frac{y-1}{3} &= 2 \\ x - 3y &= 20 \end{aligned}$
44. $\begin{aligned} \frac{x_1+4}{3} + \frac{x_2+1}{2} &= 1 \\ 3x_1 - x_2 &= -2 \end{aligned}$
45. $\begin{aligned} 0.02x_1 - 0.05x_2 &= -0.19 \\ 0.03x_1 + 0.04x_2 &= 0.52 \end{aligned}$
46. $\begin{aligned} 0.05x_1 - 0.03x_2 &= 0.21 \\ 0.07x_1 + 0.02x_2 &= 0.17 \end{aligned}$
47. $\begin{aligned} x - y - z &= 0 \\ x + 2y - z &= 6 \\ 2x - z &= 5 \end{aligned}$
48. $\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$
49. $\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 1 \\ x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - 3x_2 + 6x_3 &= 8 \end{aligned}$

50. $\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$

51. $\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$

52. $\begin{aligned} x_1 + 4x_3 &= 13 \\ 4x_1 - 2x_2 + x_3 &= 7 \\ 2x_1 - 2x_2 - 7x_3 &= -19 \end{aligned}$

53. $\begin{aligned} x - 3y + 2z &= 18 \\ 5x - 15y + 10z &= 18 \end{aligned}$

54. $\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -2 \end{aligned}$

55. $\begin{aligned} x + y + z + w &= 6 \\ 2x + 3y - w &= 0 \\ -3x + 4y + z + 2w &= 4 \\ x + 2y - z + w &= 0 \end{aligned}$

56. $\begin{aligned} -x_1 + 2x_4 &= 1 \\ 4x_2 - x_3 - x_4 &= 2 \\ x_2 - x_4 &= 0 \\ 3x_1 - 2x_2 + 3x_3 &= 4 \end{aligned}$

System of Linear Equations **In Exercises 57–62, use a software program or a graphing utility to solve the system of linear equations.**

57. $\begin{aligned} 123.5x + 61.3y - 32.4z &= -262.74 \\ 54.7x - 45.6y + 98.2z &= 197.4 \\ 42.4x - 89.3y + 12.9z &= 33.66 \end{aligned}$

58. $\begin{aligned} 120.2x + 62.4y - 36.5z &= 258.64 \\ 56.8x - 42.8y + 27.3z &= -71.44 \\ 88.1x + 72.5y - 28.5z &= 225.88 \end{aligned}$

59. $\begin{aligned} x_1 + 0.5x_2 + 0.33x_3 + 0.25x_4 &= 1.1 \\ 0.5x_1 + 0.33x_2 + 0.25x_3 + 0.21x_4 &= 1.2 \\ 0.33x_1 + 0.25x_2 + 0.2x_3 + 0.17x_4 &= 1.3 \\ 0.25x_1 + 0.2x_2 + 0.17x_3 + 0.14x_4 &= 1.4 \end{aligned}$

60. $\begin{aligned} 0.1x - 2.5y + 1.2z - 0.75w &= 108 \\ 2.4x + 1.5y - 1.8z + 0.25w &= -81 \\ 0.4x - 3.2y + 1.6z - 1.4w &= 148.8 \\ 1.6x + 1.2y - 3.2z + 0.6w &= -143.2 \end{aligned}$

61. $\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{3}{7}x_2 + \tfrac{2}{9}x_3 &= \tfrac{349}{630} \\ \tfrac{2}{3}x_1 + \tfrac{4}{9}x_2 - \tfrac{2}{5}x_3 &= -\tfrac{19}{45} \\ \tfrac{4}{5}x_1 - \tfrac{1}{8}x_2 + \tfrac{4}{3}x_3 &= \tfrac{139}{150} \end{aligned}$

62. $\begin{aligned} \tfrac{1}{8}x - \tfrac{1}{7}y + \tfrac{1}{6}z - \tfrac{1}{5}w &= 1 \\ \tfrac{1}{7}x + \tfrac{1}{6}y - \tfrac{1}{5}z + \tfrac{1}{4}w &= 1 \\ \tfrac{1}{6}x - \tfrac{1}{5}y + \tfrac{1}{4}z - \tfrac{1}{3}w &= 1 \\ \tfrac{1}{5}x + \tfrac{1}{4}y - \tfrac{1}{3}z + \tfrac{1}{2}w &= 1 \end{aligned}$

Number of Solutions **In Exercises 63–66, state why the system of equations must have at least one solution. Then solve the system and determine whether it has exactly one solution or infinitely many solutions.**

63. $\begin{aligned} 4x + 3y + 17z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$

64. $\begin{aligned} 2x + 3y &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0 \end{aligned}$

65. $\begin{aligned} 5x + 5y - z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$

66. $\begin{aligned} 16x + 3y + z &= 0 \\ 16x + 2y - z &= 0 \end{aligned}$

67. Nutrition One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 227 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 578 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

68. Airplane Speed Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Two hours after the first plane departs, the planes are 3200 kilometers apart. Find the airspeed of each plane.

True or False? **In Exercises 69 and 70, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

69. (a) A system of one linear equation in two variables is always consistent.

(b) A system of two linear equations in three variables is always consistent.

(c) If a linear system is consistent, then it has infinitely many solutions.

70. (a) A linear system can have exactly two solutions.

(b) Two systems of linear equations are equivalent when they have the same solution set.

(c) A system of three linear equations in two variables is always inconsistent.

71. Find a system of two equations in two variables, x_1 and x_2, that has the solution set given by the parametric representation $x_1 = t$ and $x_2 = 3t - 4$, where t is any real number. Then show that the solutions to the system can also be written as

$$x_1 = \frac{4}{3} + \frac{t}{3} \quad \text{and} \quad x_2 = t.$$

The symbol indicates that electronic data sets for these exercises are available at *LarsonLinearAlgebra.com*. The data sets are compatible with MATLAB, *Mathematica*, *Maple*, TI-83 Plus, TI-84 Plus, TI-89, and Voyage 200.

72. Find a system of two equations in three variables, x_1, x_2, and x_3, that has the solution set given by the parametric representation

$$x_1 = t, \quad x_2 = s, \quad \text{and} \quad x_3 = 3 + s - t$$

where s and t are any real numbers. Then show that the solutions to the system can also be written as

$$x_1 = 3 + s - t, \quad x_2 = s, \quad \text{and} \quad x_3 = t.$$

Substitution In Exercises 73–76, solve the system of equations by first letting $A = 1/x$, $B = 1/y$, and $C = 1/z$.

73. $$\begin{aligned} \frac{12}{x} - \frac{12}{y} &= 7 \\ \frac{3}{x} + \frac{4}{y} &= 0 \end{aligned}$$

74. $$\begin{aligned} \frac{3}{x} + \frac{2}{y} &= -1 \\ \frac{2}{x} - \frac{3}{y} &= -\frac{17}{6} \end{aligned}$$

75. $$\begin{aligned} \frac{2}{x} + \frac{1}{y} - \frac{3}{z} &= 4 \\ \frac{4}{x} \phantom{+ \frac{1}{y}} + \frac{2}{z} &= 10 \\ -\frac{2}{x} + \frac{3}{y} - \frac{13}{z} &= -8 \end{aligned}$$

76. $$\begin{aligned} \frac{2}{x} + \frac{1}{y} - \frac{2}{z} &= 5 \\ \frac{3}{x} - \frac{4}{y} \phantom{- \frac{2}{z}} &= -1 \\ \frac{2}{x} + \frac{1}{y} + \frac{3}{z} &= 0 \end{aligned}$$

Trigonometric Coefficients In Exercises 77 and 78, solve the system of linear equations for x and y.

77. $$\begin{aligned} (\cos\theta)x + (\sin\theta)y &= 1 \\ (-\sin\theta)x + (\cos\theta)y &= 0 \end{aligned}$$

78. $$\begin{aligned} (\cos\theta)x + (\sin\theta)y &= 1 \\ (-\sin\theta)x + (\cos\theta)y &= 1 \end{aligned}$$

Coefficient Design In Exercises 79–84, determine the value(s) of k such that the system of linear equations has the indicated number of solutions.

79. No solution
$$\begin{aligned} x + ky &= 2 \\ kx + y &= 4 \end{aligned}$$

80. Exactly one solution
$$\begin{aligned} x + ky &= 0 \\ kx + y &= 0 \end{aligned}$$

81. Exactly one solution
$$\begin{aligned} kx + 2ky + 3kz &= 4k \\ x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

82. No solution
$$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 8z &= 4 \end{aligned}$$

83. Infinitely many solutions
$$\begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

84. Infinitely many solutions
$$\begin{aligned} kx + y &= 16 \\ 3x - 4y &= -64 \end{aligned}$$

85. Determine the values of k such that the system of linear equations does not have a unique solution.
$$\begin{aligned} x + y + kz &= 3 \\ x + ky + z &= 2 \\ kx + y + z &= 1 \end{aligned}$$

86. CAPSTONE Find values of a, b, and c such that the system of linear equations has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution. Explain.
$$\begin{aligned} x + 5y + z &= 0 \\ x + 6y - z &= 0 \\ 2x + ay + bz &= c \end{aligned}$$

87. Writing Consider the system of linear equations in x and y.
$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$
Describe the graphs of these three equations in the xy-plane when the system has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution.

88. Writing Explain why the system of linear equations in Exercise 87 must be consistent when the constant terms c_1, c_2, and c_3 are all zero.

89. Show that if $ax^2 + bx + c = 0$ for all x, then $a = b = c = 0$.

90. Consider the system of linear equations in x and y.
$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$
Under what conditions will the system have exactly one solution?

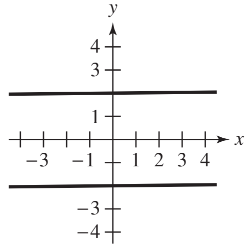
Discovery In Exercises 91 and 92, sketch the lines represented by the system of equations. Then use Gaussian elimination to solve the system. At each step of the elimination process, sketch the corresponding lines. What do you observe about the lines?

91. $$\begin{aligned} x - 4y &= -3 \\ 5x - 6y &= 13 \end{aligned}$$

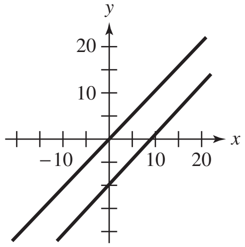
92. $$\begin{aligned} 2x - 3y &= 7 \\ -4x + 6y &= -14 \end{aligned}$$

Writing In Exercises 93 and 94, the graphs of the two equations appear to be parallel. Solve the system of equations algebraically. Explain why the graphs are misleading.

93. $$\begin{aligned} 100y - x &= 200 \\ 99y - x &= -198 \end{aligned}$$



94. $$\begin{aligned} 21x - 20y &= 0 \\ 13x - 12y &= 120 \end{aligned}$$



1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Determine the size of a matrix and write an augmented or coefficient matrix from a system of linear equations.
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations.
- Solve a homogeneous system of linear equations.

MATRICES

Section 1.1 introduced Gaussian elimination as a procedure for solving a system of linear equations. In this section, you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a **matrix.**

REMARK

The plural of matrix is *matrices*. When each entry of a matrix is a *real* number, the matrix is a **real matrix.** Unless stated otherwise, assume all matrices in this text are real matrices.

Definition of a Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \dots & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry a_{ij} is located in the ith row and the jth column. The index i is called the **row subscript** because it identifies the row in which the entry lies, and the index j is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with m rows and n columns is of **size** $m \times n$. When $m = n$, the matrix is **square** of **order** n and the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are the **main diagonal** entries.

EXAMPLE 1 Sizes of Matrices

Each matrix has the indicated size.

a. $[2]$ Size: 1×1 **b.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Size: 2×2 **c.** $\begin{bmatrix} e & 2 & -7 \\ \pi & \sqrt{2} & 4 \end{bmatrix}$ Size: 2×3

REMARK

Begin by aligning the variables in the equations vertically. Use 0 to show coefficients of zero in the matrix. Note the fourth column of constant terms in the augmented matrix.

One common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is the **augmented matrix** of the system. The matrix containing only the coefficients of the system is the **coefficient matrix** of the system. Here is an example.

System

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \qquad - 4z &= 6 \end{aligned}$$

Augmented Matrix

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$$

Cofficient Matrix

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

In the previous section, you studied three operations that produce equivalent systems of linear equations.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a finite sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are relatively simple to perform, they can involve a lot of arithmetic, so it is easy to make a mistake. Noting the elementary row operations performed in each step can make checking your work easier.

Solving some systems involves many steps, so it is helpful to use a shorthand method of notation to keep track of each elementary row operation you perform. The next example introduces this notation.

TECHNOLOGY

Many graphing utilities and software programs can perform elementary row operations on matrices. If you use a graphing utility, you may see something similar to the screen below for Example 2(c). The **Technology Guide** at *CengageBrain.com* can help you use technology to perform elementary row operations.

```
A
        [[1  2   -4  3 ]
         [0  3   -2  -1]
         [2  1   5   -2]]

mRAdd(-2,A,1,3)
        [[1  2   -4  3 ]
         [0  3   -2  -1]
         [0  -3  13  -8]]
```

EXAMPLE 2 Elementary Row Operations

a. Interchange the first and second rows.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

Notation

$R_1 \leftrightarrow R_2$

b. Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

Notation

$\left(\frac{1}{2}\right)R_1 \rightarrow R_1$

c. Add -2 times the first row to the third row to produce a new third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

Notation

$R_3 + (-2)R_1 \rightarrow R_3$

Notice that adding -2 times row 1 to row 3 does not change row 1.

In Example 7 in Section 1.1, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

EXAMPLE 3 Using Elementary Row Operations to Solve a System

Linear System

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Add the first row to the second row to produce a new second row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad R_2 + R_1 \rightarrow R_2$$

Add -2 times the first equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Add -2 times the first row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \quad R_3 + (-2)R_1 \rightarrow R_3$$

Add the second equation to the third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Add the second row to the third row to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad R_3 + R_2 \rightarrow R_3$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Use back-substitution to find the solution, as in Example 6 in Section 1.1. The solution is $x = 1$, $y = -1$, and $z = 2$.

REMARK

The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix.

The last matrix in Example 3 is in **row-echelon** form. To be in this form, a matrix must have the properties listed below.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the properties below.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

EXAMPLE 4 **Row-Echelon Form**

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is also in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

SOLUTION

The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not 1.

TECHNOLOGY

Use a graphing utility or a software program to find the row-echelon forms of the matrices in Examples 4(b) and 4(e) and the reduced row-echelon forms of the matrices in Examples 4(a), 4(b), 4(c), and 4(e). The **Technology Guide** at *CengageBrain.com* can help you use technology to find the row-echelon and reduced row-echelon forms of a matrix. Similar exercises and projects are also available on the website.

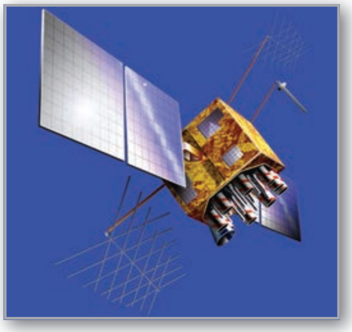
Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4(e), multiplying the second row in the matrix by $\frac{1}{2}$ changes the matrix to row-echelon form.

The procedure for using Gaussian elimination with back-substitution is summarized below.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which you perform the elementary row operations is important. Operate from *left to right by columns*, using elementary row operations to obtain zeros in all entries directly below the leading 1's.



LINEAR ALGEBRA APPLIED

The Global Positioning System (GPS) is a network of 24 satellites originally developed by the U.S. military as a navigational tool. Today, GPS technology is used in a wide variety of civilian applications, such as package delivery, farming, mining, surveying, construction, banking, weather forecasting, and disaster relief. A GPS receiver works by using satellite readings to calculate its location. In three dimensions, the receiver uses signals from at least four satellites to "trilaterate" its position. In a simplified mathematical model, a system of three linear equations in four unknowns (three dimensions and time) is used to determine the coordinates of the receiver as functions of time.

edobric/Shutterstock.com

EXAMPLE 5 **Gaussian Elimination with Back-Substitution**

Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \quad\quad &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Interchange the first two rows. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Adding -2 times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

Adding -1 times the first row to the fourth row produces a new fourth row. $R_4 + (-1)R_1 \rightarrow R_4$

Now that the first column is in the desired form, change the second column as shown below.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$

Adding 6 times the second row to the fourth row produces a new fourth row. $R_4 + (6)R_2 \rightarrow R_4$

To write the third and fourth columns in proper form, multiply the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiplying the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$ produces new third and fourth rows. $\left(\frac{1}{3}\right)R_3 \rightarrow R_3$ $\left(-\frac{1}{13}\right)R_4 \rightarrow R_4$

The matrix is now in row-echelon form, and the corresponding system is shown below.

$$\begin{aligned} x_1 + 2x_2 - x_3 \quad\quad &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Use back-substitution to find that the solution is $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, and $x_4 = 3$.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then it is unnecessary to continue the process. Simply conclude that the system has no solution, or is *inconsistent*.

EXAMPLE 6 A System with No Solution

Solve the system.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \qquad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_4 + (-3)R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. To see why this is true, convert back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

The third equation is not possible, so the system has no solution.

GAUSS-JORDAN ELIMINATION

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. Example 7 demonstrates this procedure.

EXAMPLE 7 Gauss-Jordan Elimination

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION

In Example 3, you used Gaussian elimination to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as shown below.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_2 + (-3)R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (-9)R_3 \rightarrow R_1$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2. \end{aligned}$$

The elimination procedures described in this section can sometimes result in fractional coefficients. For example, in the elimination procedure for the system

$$\begin{aligned} 2x - 5y + 5z &= 14 \\ 3x - 2y + 3z &= 9 \\ -3x + 4y &= -18 \end{aligned}$$

you may be inclined to first multiply Row 1 by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. Sometimes, judiciously choosing which elementary row operations you apply, and the order in which you apply them, enables you to avoid fractions.

REMARK

No matter which elementary row operations or order you use, the reduced row-echelon form of a matrix is the same.

DISCOVERY

1. Without performing any row operations, explain why the system of linear equations below is consistent.

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ -5x_1 + 6x_2 - 17x_3 &= 0 \\ 7x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

2. The system below has more variables than equations. Why does it have an infinite number of solutions?

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 &= 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with infinitely many solutions.

EXAMPLE 8 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}.$$

Using a graphing utility, a software program, or Gauss-Jordan elimination, verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter t to represent x_3, you have

$$x_1 = 2 - 5t, \quad x_2 = -1 + 3t, \quad x_3 = t, \quad t \text{ is any real number.}$$

Note in Example 8 that the arbitrary parameter t represents the *nonleading* variable x_3. The variables x_1 and x_2 are written as functions of t.

You have looked at two elimination methods for solving a system of linear equations. Which is better? To some degree the answer depends on personal preference. In real-life applications of linear algebra, systems of linear equations are usually solved by computer. Most software uses a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data. The examples and exercises in this text focus on the underlying concepts, so you should know both elimination methods.

HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations in which each of the constant terms is zero are called **homogeneous.** A homogeneous system of m equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

REMARK

A homogeneous system of three equations in the three variables x_1, x_2, and x_3 has the trivial solution $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$.

A homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations is satisfied. Such a solution is **trivial** (or obvious).

EXAMPLE 9 Solving a Homogeneous System of Linear Equations

Solve the system of linear equations.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

SOLUTION

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

yields the matrices shown below.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \qquad R_2 + (-2)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad \left(\tfrac{1}{3}\right)R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad R_1 + R_2 \rightarrow R_1$$

The system of equations corresponding to this matrix is

$$\begin{aligned} x_1 \qquad + 2x_3 &= 0 \\ x_2 - x_3 &= 0. \end{aligned}$$

Using the parameter $t = x_3$, the solution set is $x_1 = -2t$, $x_2 = t$, and $x_3 = t$, where t is any real number. This system has infinitely many solutions, one of which is the trivial solution ($t = 0$).

As illustrated in Example 9, a homogeneous system with fewer equations than variables has infinitely many solutions.

THEOREM 1.1 The Number of Solutions of a Homogeneous System

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

To prove Theorem 1.1, use the procedure in Example 9, but for a general matrix.

1.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.

Matrix Size **In Exercises 1–6, determine the size of the matrix.**

1. $\begin{bmatrix} 1 & 2 & -4 \\ 3 & -4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & -1 & 1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -1 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 3 & 4 & -10 \end{bmatrix}$

Elementary Row Operations **In Exercises 7–10, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.**

7. *Original Matrix* $\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$ *New Row-Equivalent Matrix* $\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$

8. *Original Matrix* $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$ *New Row-Equivalent Matrix* $\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$

9. *Original Matrix* $\begin{bmatrix} 0 & -1 & -7 & 7 \\ -1 & 5 & -8 & 7 \\ 3 & -2 & 1 & 2 \end{bmatrix}$ *New Row-Equivalent Matrix* $\begin{bmatrix} -1 & 5 & -8 & 7 \\ 0 & -1 & -7 & 7 \\ 0 & 13 & -23 & 23 \end{bmatrix}$

10. *Original Matrix* $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$ *New Row-Equivalent Matrix* $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

Augmented Matrix **In Exercises 11–18, find the solution set of the system of linear equations represented by the augmented matrix.**

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 3 & -1 & 1 & 5 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Row-Echelon Form **In Exercises 19–24, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.**

19. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

21. $\begin{bmatrix} -2 & 0 & 1 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

System of Linear Equations **In Exercises 25–38, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.**

25. $\begin{aligned} x + 3y &= 11 \\ 3x + y &= 9 \end{aligned}$

26. $\begin{aligned} 2x + 6y &= 16 \\ -2x - 6y &= -16 \end{aligned}$

27. $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

28. $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

29. $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

30. $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$